

The Hamiltonian approach in classification and integrability of hydrodynamic chains

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Abstract

New approach in classification of integrable hydrodynamic chains is established. This is the method of the *Hamiltonian* hydrodynamic reductions. Simultaneously, this approach yields *explicit* Hamiltonian hydrodynamic reductions of the Hamiltonian hydrodynamic chains. The concept of *reducible* Poisson brackets is established. Also this approach is useful for non-Hamiltonian hydrodynamic chains. The *deformed* Benney hydrodynamic chain is considered.

Contents

1	Introduction	2
2	The Hamiltonian approach for 2+1 quasilinear equations	4
3	<i>Reducible</i> Poisson brackets and hydrodynamic chains	7
4	The integrability criteria	12
5	Classification of simplest Hamiltonian hydrodynamic chains	16
5.1	The first integrability criterion (extra conservation law)	16
5.2	The second integrability criterion (generating functions of conservation laws)	18
5.3	The third integrability criterion (the Hamiltonian hydrodynamic reductions)	20
6	Hydrodynamic chains associated with the Kupershmidt brackets	22
6.1	The <i>linear</i> Hamiltonian density	24
6.2	The Miura type and reciprocal transformations	25
6.3	Reciprocal transformations and generating functions of conservation laws .	26
7	Hydrodynamic chains determined by the Kupershmidt–Manin bracket	28
7.1	The Benney hydrodynamic chain	29
7.2	The <i>mixed</i> modified Benney hydrodynamic chain	29
7.3	Reciprocal transformations and non- <i>linear</i> cases	31
8	Non-Hamiltonian hydrodynamic chains	32
9	The Zakharov hydrodynamic reductions	34

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1 Introduction

This paper is devoted to classification and integrability of 2+1 quasilinear equations and corresponding hydrodynamic chains (see [10], [27], [28]) by the “Hamiltonian” approach based on the Hamiltonian formulation of integrable hydrodynamic chains and their hydrodynamic reductions. Nevertheless, this Hamiltonian approach is much more universal, because most of explicit hydrodynamic reductions are non-Hamiltonian and this approach is useful as well as for non-Hamiltonian hydrodynamic chains. However, without loss of generality an illustration of this approach will be given on important examples connected with so-called M -Poisson brackets (see details in [30]).

A hydrodynamic type system is N component quasilinear system of PDE’s of the first order (see [5])

$$u_t^i = v_j^i(\mathbf{u})u_x^j, \quad i, j = 1, 2, \dots, N. \quad (1)$$

A hydrodynamic chain is a generalization of a hydrodynamic type system on an infinite component case (see details below). At the same time, a hierarchy of commuting hydrodynamic chains is equivalent to a family of 2+1 integrable quasilinear equations (see [7], [27], [28]). Hydrodynamic chains and corresponding 2+1 quasilinear equations can be decomposed in infinitely many ways on families of 1+1 integrable hydrodynamic type systems (1) (see [7], [16], [28]). Nonlinear PDE’s describing such hydrodynamic reductions have solutions parameterized by N arbitrary functions of a single variable (see [7], [16]). These nonlinear PDE’s are still not solved yet (except [28]), but infinitely many particular *explicit* solutions parameterized by an arbitrary number of constants been known for the Benney hierarchy many years ago (see [17], [18], [20], [22], [37]). Moreover, new hydrodynamic reductions can be found by this Hamiltonian approach (see, for instance, [29] and [33]). Every hydrodynamic reduction determines some particular solution of corresponding 2+1 quasilinear equation. If such an equation is homogeneous, then corresponding solution is self-similar. Such solutions can be found by different methods (see [25]), based on the twistor approach, theory of the Beltrami and the Hamilton-Jacobi equations, the conformal mapping and the dispersionless $\bar{\partial}$ -method, differential and algebraic geometry (see also [14]). An application of the generalized hodograph method yields most general non-self similar solutions by the nonlinear superposition principle (see [35]). For instance, the “symmetry constraint” method established in [3] comes from a theory of 2+1 integrable dispersive nonlinear PDE’s such as the Kadomtsev–Petviashvili

(KP) or the Veselov–Novikov (VN) equations. The Hamiltonian approach is much more *universal*, because this method exists *irrespective of origin* of given hydrodynamic chain.

The problem of classification of Hamiltonian chains consists of two steps. This Hamiltonian approach is used to suggest some list of new Poisson brackets connected with hydrodynamic chains. Such classification of the Poisson brackets should be done in further publications (see also [30]), at least all “reducible” Poisson brackets (see definition below) should be described (nevertheless, we believe that any Poisson bracket is reducible by some appropriate “moment decomposition”). Such *linear* Poisson brackets have been introduced by I. Dorfman in [4]. The **main claim** of this paper is that *the Hamiltonian structure completely determines hierarchy of integrable hydrodynamic chains up to the Miura type and reciprocal transformations* (This is not a true for integrable dispersive and hydrodynamic type systems). Thus, a description of the Hamiltonian structures means a description of integrable hydrodynamic chains. In this paper without loss of generality we restrict our consideration on the Hamiltonian densities depending on first three moments A^k . Obviously, they can be *linear* $\mathbf{H} = A^2 + f(A^0, A^1)$, *quasilinear* $\mathbf{H} = g(A^0, A^1)A^2 + f(A^0, A^1)$ and *fully nonlinear* $\mathbf{H}(A^0, A^1, A^2)$. Since the integrable hydrodynamic chains determined by the *quasilinear* and *fully nonlinear* Hamiltonian densities can be reduced (by some appropriate reciprocal transformations) to the integrable hydrodynamic chains determined by the *linear* Hamiltonian densities, we restrict our consideration on the *linear* case in this paper only. Two others case are just briefly described.

The paper is organized in the following order. In the second section the method allowing to extract explicit Hamiltonian hydrodynamic reductions with the aid of pseudopotentials is established. In the third section several *reducible* multi-parametric families of the Poisson brackets are derived. In the fourth section the integrability criteria of hydrodynamic chains are considered. In the fifth section the first such a family of the Poisson brackets is investigated by three different methods. Corresponding integrable hydrodynamic chains are found by the requirement of the existence of extra conservation law creating a commuting flow (see [21], [27]), by the method of pseudopotentials (see [7], [36]) and separately by an application of the *phenomenological* Hamiltonian approach. All these hydrodynamic chains are connected with integrable hydrodynamic chains from the sixth section by a reciprocal transformation. In the sixth section hydrodynamic chains associated with the Kupershmidt Poisson brackets are considered. In the seventh section a classification of integrable Hamiltonian hydrodynamic chains associated with the Kupershmidt–Manin Poisson bracket is significantly simplified for the *linear* Hamiltonian density (cf. [10]). In the eighth section the method of Hamiltonian hydrodynamic reductions is extended on non-Hamiltonian hydrodynamic chains. Corresponding hydrodynamic type systems are non-Hamiltonian too. In the ninth section the modified Kupershmidt hydrodynamic chain is investigated with the aid of the Zakharov hydrodynamic reductions.

2 The Hamiltonian approach for 2+1 quasilinear equations

The “Hamiltonian” approach is universal and based on the “concept” of pseudo-potentials (see [7], [27], [36]). Let us briefly describe and illustrate this approach on the famous Benney hydrodynamic chain [1]

$$A_t^k = A_x^{k+1} + kA^{k-1}A_x^0, \quad k = 0, 1, 2, \dots \quad (2)$$

and corresponding the Khohlov–Zabolotzkaya equation

$$(A_y^0 - A^0 A_x^0)_x = A_{tt}^0,$$

which can be obtained with the aid of two first equations from the Benney hydrodynamic chain and the first equation from its first commuting flow

$$A_y^k = A_x^{k+2} + A^0 A_x^k + (k+1)A^k A_x^0 + kA^{k-1}A_x^1, \quad k = 0, 1, 2, \dots$$

by eliminating of the moments A^1 and A^2 . The method of hydrodynamic reductions developed for these equations (see [16]) was extended for more wide class of 2+1 hydrodynamic type systems (see [7]), more general 2+1 quasilinear equations and hydrodynamic chains (see [8] and [9]) as an integrability criterion. The above equations have N component hydrodynamic reductions written in the Riemann invariants

$$r_t^i = \mu^i(\mathbf{r})r_x^i, \quad r_y^i = (\mu^{i^2}(\mathbf{r}) + A^0(\mathbf{r}))r_x^i, \quad i = 1, 2, \dots, N \quad (3)$$

consistent with the generating functions of conservation laws

$$\mu_t = \partial_x \left(\frac{\mu^2}{2} + A^0 \right), \quad \mu_y = \partial_x \left(\frac{\mu^3}{3} + A^0 \mu + A^1 \right), \quad (4)$$

where all moments A^k are some functions of the Riemann invariants r^n . Following [16], the Gibbons–Tsarev system

$$\partial_i \mu^k = \frac{\partial_i A^0}{\mu^i - \mu^k}, \quad \partial_{ik} A^0 = 2 \frac{\partial_i A^0 \partial_k A^0}{(\mu^i - \mu^k)^2}, \quad i \neq k \quad (5)$$

can be obtained from the compatibility conditions $\partial_k(\partial_i \mu) = \partial_i(\partial_k \mu)$, where

$$\partial_i \mu = \frac{\partial_i A^0}{\mu^i - \mu}, \quad (6)$$

which is a consequence that μ is a conservation law density of the hydrodynamic type systems (3). Taking into account all moments A^k are connected with the Benney hydrodynamic chain by the Laurent series

$$\lambda = \mu + \frac{A^0}{\mu} + \frac{A^1}{\mu^2} + \frac{A^2}{\mu^3} + \dots \quad (7)$$

one can verify that the Benney hydrodynamic chain together with any its hydrodynamic reductions satisfy the Gibbons equation (see [15])

$$\lambda_t - \mu \lambda_x = \frac{\partial \lambda}{\partial \mu} [\mu_t - \partial_x (\frac{\mu^2}{2} + A^0)]. \quad (8)$$

We believe that the Gibbons–Tsarev system is Darboux integrable, because plenty particular solutions parameterized by arbitrary constants are known. However, a general solution still is unknown.

In this paper the Hamiltonian approach is established. Let us choose N arbitrary conservation law densities $\mu^{(k)} = a^k$, then

$$a_t^k = \partial_x \left(\frac{(a^k)^2}{2} + A^0(\mathbf{u}) \right). \quad (9)$$

The transformation $a^k(\mathbf{r})$ is invertible. Then the Gibbons–Tsarev system (in the independent variables a^k on sole function A^0 only, cf. (5))

$$(a^i - a^k) \partial_{ik} A^0 = \partial_k A^0 \partial_i \left(\sum \partial_n A^0 \right) - \partial_i A^0 \partial_k \left(\sum \partial_n A^0 \right), \quad i \neq k, \quad (10)$$

$$(a^i - a^k) \frac{\partial_{ik} A^0}{\partial_i A^0 \partial_k A^0} + (a^k - a^j) \frac{\partial_{jk} A^0}{\partial_j A^0 \partial_k A^0} + (a^j - a^i) \frac{\partial_{ij} A^0}{\partial_i A^0 \partial_j A^0} = 0, \quad i \neq j \neq k,$$

can be derived from the compatibility conditions (cf. (6)) $\partial_k(\partial_i \mu) = \partial_i(\partial_k \mu)$, where $\partial_i \equiv \partial/\partial a^i$ and

$$\partial_i \mu = \frac{\partial_i A^0}{\mu - a^i} \left[\sum \frac{\partial_m A^0}{\mu - a^m} - 1 \right]^{-1}. \quad (11)$$

Assume a^k are flat coordinates (see [5]), then the corresponding Poisson bracket can be reduced (by linear transformation of field variables a^k) to the diagonal form

$$\{a^i, a^j\} = (\varepsilon_i)^{-1} \delta^{ij} \delta'(x - x'), \quad (12)$$

where δ^{ij} is the Kronecker symbol.

Theorem 1: *The hydrodynamic type system (9) with the local Hamiltonian structure*

$$a_t^k = \frac{1}{\varepsilon_k} \partial_x \frac{\partial \mathbf{h}_2}{\partial a^k} \quad (13)$$

is a hydrodynamic reduction of the Benney hydrodynamic chain iff $A^0 = \sum \varepsilon_k a^k$ and the Hamiltonian density $\mathbf{h}_2 = \sum \varepsilon_k (a^k)^3/6 + (A^0)^2/2$.

Proof: Let us substitute the expression $A^0 = \sum \varepsilon_k a^k$ in (11). This ODE system

$$\partial_i \mu = \frac{\varepsilon_i}{\mu - a^i} \left[\sum \frac{\varepsilon_m}{\mu - a^m} - 1 \right]^{-1}$$

can be integrated in the implicit form

$$\lambda = \mu - \sum \varepsilon_k \ln(\mu - a^k), \quad (14)$$

where λ is an integration factor. Simultaneously, the equation of the Riemann surface satisfies the Gibbons equation (8).

Corollary: Since $A^0 = \Sigma \varepsilon_k a^k$, then all other moments

$$A^n = \frac{1}{n+1} \sum \varepsilon_k (a^k)^{n+1}, \quad (15)$$

if $\Sigma \varepsilon_k = 0$. It is easy to prove by comparison (2) with (9). Then the Laurent series (7) reduces to so-called “waterbag” reduction (see [17]) for the equation of the Riemann surface (14). If $\Sigma \varepsilon_k \neq 0$ the above formula is valid (see [19]), but expressions of the higher moment A^n can be obtained by the substitution of the Laurent series (7) into the “deformed” above equation

$$\lambda - \sum \varepsilon_k \ln \lambda = \mu - \sum \varepsilon_k \ln(\mu - a^k),$$

because the function λ can be replaced by an arbitrary function $\tilde{\lambda} = \tilde{\lambda}(\lambda)$ (see (8)).

Remark: The famous Zakharov reduction (see [37])

$$a_t^k = \partial_x \left(\frac{(a^k)^2}{2} + A^0 \right), \quad b_t^k = \partial_x (u^k b^k), \quad A^0 = \sum b^m \quad (16)$$

can be obtained in the same way. In comparison with the above “waterbag” case, let us expand the Taylor series at the vicinity of the local parameter λ

$$\mu^{(i)} = a^i + \lambda b^i + \lambda^2 c^i + \dots$$

If first $2N$ conservation law densities a^i and b^i are independent, then the above hydrodynamic reduction (without the fixation $A^0 = \Sigma b^n$) is given by the moment decomposition

$$A^k = \sum (a^i)^k b^i. \quad (17)$$

Suppose a^i and b^i are flat coordinates, then a simplest local Hamiltonian structure is

$$a_t^k = \partial_x \frac{\partial \mathbf{h}_2}{\partial b^k}, \quad b_t^k = \partial_x \frac{\partial \mathbf{h}_2}{\partial a^k}, \quad (18)$$

then the Hamiltonian density $\mathbf{h}_2 = \Sigma (a^n)^2 b^n / 2 + (A^0)^2 / 2$, where, indeed, $A^0 = \Sigma b^n$.

For the first time the concept of “pseudopotentials” was introduced for 2+1 quasilinear equation obtained by the dispersionless limit of 2+1 dispersive systems or obtained by the longwave (continuum) limit of 2+1 discrete systems in [36]. The meaning of the “pseudopotential” is so-called “dispersionless limit of the Lax formulation” for the integrable dispersive systems. This is nothing but the generating function of conservation laws for hydrodynamic chains (see the above example (4)). Pseudopotentials can be effectively found for aforementioned 2+1 hydrodynamic type systems (see [7]), for 2+1 quasilinear equations (see [8]) and for hydrodynamic chains (see [30]). Thus,

1. we suppose pseudopotentials for the Hamiltonian hydrodynamic chains are given (cf. (4)) in the explicit form

$$\mu_t = \partial_x f(\mu; U^1, U^2, \dots, U^M),$$

where the functions U^k depend on a finite number of first moments A^k ;

2. we look for explicit hydrodynamic reductions in the Hamiltonian form (13), where

$$d\mathbf{h} = \sum_{k=1}^N \varepsilon_k f(a^k; U^1(\mathbf{a}), U^2(\mathbf{a}), \dots, U^M(\mathbf{a})) d\mu^k.$$

The existence of the Hamiltonian density \mathbf{h} means the existence of *local* Hamiltonian hydrodynamic reductions. The same procedure can be used for more general local Hamiltonian structures (see [5]) than (13)

$$a_t^i = \partial_x [\bar{g}^{ik} \frac{\partial \mathbf{h}}{\partial a^k}], \quad i = 1, 2, \dots, N, \quad (19)$$

where \bar{g}^{ij} is a constant, non-degenerate and symmetric matrix, and for *nonlocal* Hamiltonian hydrodynamic reductions (see [6]).

Conjecture 1: *Any integrable hydrodynamic chain (and associated 2+1 quasilinear system) admits N component hydrodynamic reductions*

$$a_t^k = \partial_x f(a^k; U^1(\mathbf{a}), U^2(\mathbf{a}), \dots, U^M(\mathbf{a}))$$

written in the Hamiltonian form (19).

Below we reformulate this Hamiltonian approach in the *reverse* direction: if the hydrodynamic chain has the Hamiltonian structure, then their explicit hydrodynamic reductions are the Hamiltonian hydrodynamic type systems.

3 Reducible Poisson brackets and hydrodynamic chains

The **main observation** successfully utilized in this paper is that the first local Hamiltonian structure (18) of the Zakharov reduction (16) of the Benney hydrodynamic chain (2) can be used for a *constructing* the first local Hamiltonian structure (see [22])

$$A_t^k = \{A^k, \bar{\mathbf{H}}_2\} = [kA^{k+n-1}\partial_x + n\partial_x A^{k+n-1}] \frac{\delta \bar{\mathbf{H}}_2}{\delta A^n}, \quad (20)$$

where (see (17)) the moments $A^k = \Sigma(a^m)^k b^m$. The Kupershmidt–Manin bracket (see (20))

$$\{A^k, A^n\} = [kA^{k+n-1}\partial_x + n\partial_x A^{k+n-1}]\delta(x - x'), \quad k, n = 0, 1, 2, \dots \quad (21)$$

reduces to the *canonical* Poisson bracket (see (18) and details in [5], just non-zero component below)

$$\{a^k, b^k\} = \delta'(x - x') \quad (22)$$

under the above moment decomposition $A^k = \Sigma(a^m)^k b^m$. The Benney hydrodynamic chain (2) is determined by the Hamiltonian $\bar{\mathbf{H}}_2 = \frac{1}{2} \int [A^2 + (A^0)^2] dx$, while the Kupershmidt–Manin bracket is associated with the momentum $\bar{\mathbf{H}}_1 = \int A^1 dx$ and the Casimir (annihilator) $\bar{\mathbf{H}}_0 = \int A^0 dx$. The Zakharov reduction of the Benney hydrodynamic chain (16) is determined by the Hamiltonian $\bar{\mathbf{h}}_2 = \frac{1}{2} \int [\Sigma(a^k)^2 b^k + (\Sigma b^k)^2] dx$, the momentum $\bar{\mathbf{h}}_1 = \int \Sigma a^k b^k dx$ and $2N$ Casimirs $\bar{\mathbf{h}}_{(k)} = \int b^k dx$, $\bar{\mathbf{h}}_{(N+k)} = \int a^k dx$.

In general case the local Hamiltonian structure for the hydrodynamic type system (1)

$$u_t^i = \{u^i, \bar{\mathbf{h}}\} = [g^{ij}\partial_x - g^{is}\Gamma_{sk}^j u_x^k] \frac{\delta \bar{\mathbf{h}}}{\delta u^j}, \quad (23)$$

is determined by the Hamiltonian $\bar{\mathbf{h}} = \int h(\mathbf{u})dx$ and by the Dubrovin–Novikov bracket (a differential-geometric Poisson bracket of the first order, see [5])

$$\{u^i(x), u^j(x')\} = [g^{ij}\partial_x - g^{is}\Gamma_{sk}^j u_x^k] \delta(x - x'), \quad i, j = 1, 2, \dots, N, \quad (24)$$

where a **flat** metric $g^{ij}(\mathbf{u})$ is symmetric and non-degenerate, $\Gamma_{sk}^j = \frac{1}{2}g^{jm}(\partial_s g_{mk} + \partial_k g_{ms} - \partial_m g_{sk})$ are the Christoffel symbols. Such local Hamiltonian structure can be written via so-called the *Liouville* coordinates $A^k(\mathbf{u})$, that the corresponding Poisson bracket is

$$\{A^k(x), A^n(x')\} = [\mathcal{W}^{kn}(\mathbf{A})\partial_x + \partial_x \mathcal{W}^{nk}(\mathbf{A})] \delta(x - x'), \quad k, n = 1, 2, \dots, N.$$

Formally, this Poisson bracket can be extended on an infinite component case, see the example (21). *The **main** problem in a classification of integrable hydrodynamic chains is a description of such Poisson brackets*

$$\{A^k(x), A^n(x')\} = [\mathcal{W}^{kn}(\mathbf{A})\partial_x + \partial_x \mathcal{W}^{nk}(\mathbf{A})] \delta(x - x'), \quad k, n = 1, 2, \dots, \quad (25)$$

where the coefficients $\mathcal{W}^{kn}(\mathbf{A})$ satisfy the Jacobi identity (see [30])

$$\begin{aligned} (\mathcal{W}^{ik} + \mathcal{W}^{ki})\partial_k \mathcal{W}^{nj} &= (\mathcal{W}^{jk} + \mathcal{W}^{kj})\partial_k \mathcal{W}^{ni}, \\ \partial_n \mathcal{W}^{ij}\partial_m \mathcal{W}^{kn} &= \partial_n \mathcal{W}^{kj}\partial_m \mathcal{W}^{in}. \end{aligned} \quad (26)$$

Definition 1: A Poisson bracket (25) is said to be **reducible** if this Poisson bracket is equivalent to the Dubrovin–Novikov bracket (24) under the some restriction $A^k = f^k(\mathbf{u})$.

Conjecture 2: Possibly, one can find appropriate constraints $A^k = f^k(\mathbf{u})$ for any coefficients $\mathcal{W}^{kn}(\mathbf{A})$, that the Poisson bracket (25) becomes *reducible* to the Dubrovin–Novikov bracket (24) in infinitely many ways.

Example 1: The Kupershmidt–Manin bracket (21) is reducible, because at least one moment decomposition $A^k = \Sigma(a^m)^k b^m$ exists (see (17), (22)). Below $N - 1$ parametric family of a moment decomposition is described.

As usual we call the “hydrodynamic type system” the 1+1 quasilinear system of non-linear PDE’s of the first order (see (1) and details in [5]). *If $N \rightarrow \infty$, but the number of components $v_j^i(\mathbf{u})$ is finite for every index i and each of these components is a function of finite number of field variables u^k , then such hydrodynamic type systems we call “hydrodynamic chains” (see details in [10], [27], [28]). Also, we re-numerate equations of hydrodynamic chains in order that every row determined by the index i depends on a number of field variables larger than i . Thus, the hydrodynamic chain is an infinite component generalization of a hydrodynamic type system*

$$A_t^k = \sum_{n=0}^{N_k} V_n^k(\mathbf{A}) A_x^n, \quad k = 1, 2, \dots, \quad (27)$$

where $V_n^k(\mathbf{A})$ are functions of the moments A^m , $m = 1, 2, \dots, M_k$; M_k and N_k are some integers.

Let us introduce the infinite set of the moments $A^k = f^k(\mathbf{u})$, where $f^k(\mathbf{u})$ are some functions. For simplicity, below we shall describe hydrodynamic type systems (1) whose coefficients $v_j^i(\mathbf{u})$ depend explicitly on number N of field variables u^k .

Example 2: The Zakharov reduction (16) explicitly depends on number of the field variables b^k (because $A^0 = \Sigma b^k$), but the Benney hydrodynamic chain does not depend on this number.

If the hydrodynamic type system (1) can be written via these moments A^k , whose coefficients $V_j^i(\mathbf{A})$ are **independent** on the above number N , then such a hydrodynamic type system (1) we call a **hydrodynamic reduction** (see definition and details below) of the **hydrodynamic chain** (27). The variables a^k are said to be “flat” coordinates, the hydrodynamic type system (1) is said to be written in the *canonical* form (19), if the Christoffel symbols are vanished (see [5], [35]).

Let us re-compute the canonical Poisson bracket (see (19))

$$\{a^i, a^j\} = \bar{g}^{ij} \delta'(x - x') \quad (28)$$

via moments A^k :

$$\{A^k, A^n\} = \frac{\partial f^k(\mathbf{a})}{\partial a^s} \bar{g}^{sm} \partial_x \frac{\partial f^n(\mathbf{a})}{\partial a^m} \delta(x - x'). \quad (29)$$

Assume the above r.h.s. can be expressed via moments A^k only, i.e. the coefficients $\mathcal{W}^{kn}(\mathbf{A})$ in (25) are **independent** on number N (see the examples below).

Definition 2: The hydrodynamic chain (27) written in the form (25)

$$A_t^k = \{A^k, \bar{\mathbf{H}}\} = [\mathcal{W}^{kn}(\mathbf{A}) \partial_x + \partial_x \mathcal{W}^{nk}(\mathbf{A})] \frac{\delta \bar{\mathbf{H}}}{\delta A^n}, \quad k, n = 0, 1, 2, \dots \quad (30)$$

is said to be *Hamiltonian*.

Remark: The above local Hamiltonian structures (30) and corresponding Poisson brackets still are not investigated properly. However, plenty publications are devoted to their particular cases (see, for instance, [4], [21], [22]), some of them will be described below. The classification of so-called M brackets is given in [30]. All examples presented in this paper belong to this class.

Local Hamiltonian structures (30) for hydrodynamic chains we call *reducible* if they are obtained by the above procedure directly from local Hamiltonian structures (see (28)) written in flat coordinates of corresponding hydrodynamic reductions. So, in this paper we describe a broad class of hydrodynamic chains associated with reducible Poisson brackets. All other Poisson brackets (30) and corresponding integrable hydrodynamic chains will be discussed elsewhere.

Let us, for instance, introduce the infinite set of moments A^k via multi-index constant matrices ε

$$A^1 = \varepsilon_k^{(0)} a^k + \varepsilon^0, \quad A^2 = \varepsilon_{jk}^{(0)} a^j a^k + \varepsilon_k^{(1)} a^k + \varepsilon^1, \quad A^3 = \varepsilon_{ijk}^{(0)} a^i a^j a^k + \varepsilon_{jk}^{(1)} a^j a^k + \varepsilon_k^{(2)} a^k + \varepsilon^2, \dots,$$

where, obviously, all multi-index matrices ε are symmetric under a permutation of their sub-indexes. Then

$$\begin{aligned}\{A^1, A^1\} &= \frac{\partial A^0}{\partial a^i} \bar{g}^{ij} \partial_x \frac{\partial A^0}{\partial a^j} \delta(x - x') = \varepsilon_k^{(0)} \bar{g}^{km} \varepsilon_m^{(0)} \delta'(x - x'), \\ \{A^2, A^1\} &= \frac{\partial A^1}{\partial a^i} \bar{g}^{ij} \partial_x \frac{\partial A^0}{\partial a^j} \delta(x - x') = [2\varepsilon_m^{(0)} \bar{g}^{ms} \varepsilon_{sk}^{(0)} a^k + \varepsilon_m^{(1)} \bar{g}^{ms} \varepsilon_s^{(0)}] \delta'(x - x'), \\ \{A^2, A^2\} &= \frac{\partial A^1}{\partial a^i} \bar{g}^{ij} \partial_x \frac{\partial A^1}{\partial a^j} \delta(x - x') = [4\varepsilon_{ns}^{(0)} \bar{g}^{sm} \varepsilon_{mk}^{(0)} a^n a^k + 4\varepsilon_m^{(1)} \bar{g}^{ms} \varepsilon_{sk}^{(0)} a^k \\ &\quad + \varepsilon_m^{(1)} \bar{g}^{ms} \varepsilon_s^{(1)}] \delta'(x - x') + 2[\varepsilon_{ns}^{(0)} \bar{g}^{sm} \varepsilon_{mk}^{(0)} a^n a^k + \varepsilon_m^{(1)} \bar{g}^{ms} \varepsilon_{sk}^{(0)} a^k]_x \delta(x - x'), \dots\end{aligned}$$

The main problem in the classification of such local Hamiltonian structures for hydrodynamic chains is a description of all multi-index matrices ε allowing to **express explicitly** the polynomials with respect to flat coordinates a^k in right hand sides of the above Poisson bracket via **polynomials** of moments A^n . Then such moments are analogues of the Liouville coordinates for an infinite dimensional space (see [5], [24], [30]), i.e.

$$\begin{aligned}\{A^0, A^0\} &= \alpha^{00} \delta'(x - x'), \quad \{A^1, A^0\} = [\alpha^{10} A^0 + \alpha^{11}] \delta'(x - x'), \\ \{A^1, A^1\} &= [(\alpha^{20} A^1 + \alpha^{21} (A^0)^2 + \alpha^{22}) \partial_x + \partial_x (\alpha^{20} A^1 + \alpha^{21} (A^0)^2 + \alpha^{22})] \delta(x - x'), \dots\end{aligned}$$

All coefficients α^{ik} must be **independent** on N .

Example 3: Let us introduce the moments

$$A^k = \frac{1}{\beta k + \gamma} \sum_{i=1}^N \varepsilon_i (a^i)^{\beta k + \gamma}, \quad k = 0, \pm 1, \pm 2, \dots, \quad (31)$$

where ε_i , β and γ are arbitrary constants. Any symmetric non-degenerate matrix \bar{g}^{ij} can be written as a diagonal matrix under a linear change of the field variables a^i . Thus, without loss of generality we assume that diagonal metric coefficients are α_i (off-diagonal coefficients are absent, see (12)). Then we have (see (29))

$$\{A^k, A^n\} = \sum_{i=1}^N \alpha_i \varepsilon_i^2 [(a^i)^{\beta(k+n)+2(\gamma-1)} \partial_x + (\beta n + \gamma - 1) (a^i)^{\beta(k+n)+2(\gamma-1)} a_x^i] \delta(x - x').$$

In general case r.h.s. can be expressed via moments A^k in **infinitely many ways** for every *fixed* number N . However, in the above construction r.h.s. cannot depend on number N . Since A^k is a sum of monomials of the sole degree $\beta k + \gamma$, then the sum of monomials of another sole degree $\beta(k+n) + 2(\gamma-1)$ must be expressed via the moments A^{k+n+m} , where m is an integer, which is not determined yet. Then $\alpha_i = 1/\varepsilon_i$ and

$$\{A^k, A^n\} = [(\beta(k+m) + 1) A^{k+n+m} \partial_x + (\beta(n+m) + 1) \partial_x A^{k+n+m}] \delta(x - x'),$$

where $\gamma = \beta m + 2$. Since $\beta \neq 0$, one can re-scale all moments $A^k \rightarrow \beta A^k$. Introducing new parameter $l = 1/\beta$, one can obtain the two parametric family of the Poisson brackets

$$\{A_{(m,l)}^k, A_{(m,l)}^n\} = [(k+m+l) A_{(m,l)}^{k+n+m} \partial_x + (n+m+l) \partial_x A_{(m,l)}^{k+n+m}] \delta(x - x'),$$

where

$$A_{(m,l)}^k = \frac{l^2}{k+m+2l} \sum_{i=1}^N \varepsilon_i (a^i)^{\frac{k+m}{l}+2}, \quad k, m \in \mathbf{Z} \quad (32)$$

Remark: If also $l \in \mathbf{Z}$, then the some value of running index $k^* = -(m+2l)$ determines the corresponding moment

$$A_{(m,l)}^{-m-2l} = l \sum_{i=1}^N \varepsilon_i \ln a^i.$$

Remark: The above Poisson bracket can be obtained with the aid of another moment decomposition (cf. (17))

$$A_{(m,l)}^k = l \sum_{i=1}^N (a^i)^{\frac{k+m}{l}+1} b^i,$$

where another canonical Poisson bracket (22).

Example 4: Let us consider the *nonlocal* Poisson bracket (see [13])

$$\{a^i, a^j\} = \left[\left(\frac{\delta_{ij}}{\varepsilon_i} - \varepsilon a^i a^j \right) \partial_x - \varepsilon a^j a_x^i + \varepsilon a_x^i \partial_x^{-1} a_x^j \right] \delta(x - x'),$$

where δ_{ij} is the Kronecker symbol and ε is a constant curvature. Introducing the moments A^k with the aid of (31), the above Poisson bracket leads to the Poisson bracket

$$\begin{aligned} \{A^k, A^n\} &= [(\beta(k+m)+1)A^{k+n+m}\partial_x + (\beta(n+m)+1)\partial_x A^{k+n+m}] \delta(x-x') \\ &\quad - \varepsilon [(\beta(k+m)+2)(\beta(n+m)+2)A^k A^n \partial_x + (\beta(k+m)+2)(\beta(n+m)+1)A^k A_x^n \\ &\quad + (\beta(n+m)+2)A^n A_x^k - A_x^k \partial_x^{-1} A_x^n] \delta(x-x'), \end{aligned}$$

where $\gamma = \beta m + 2$.

Example 5: Let us introduce the moments (cf. (31))

$$A^k = \frac{1}{\beta k + \gamma} \sum_{i=1}^N \varepsilon_i [f(a^i)]^{\beta k + \gamma}, \quad k = 0, \pm 1, \pm 2, \dots,$$

where

$$f'^2(z) = R_M(f) \equiv \alpha_0 f^M + \alpha_1 f^{M-1} + \alpha_2 f^{M-2} + \dots + \alpha_M.$$

Then we have

$$\{A^k, A^n\} = \sum_{i=1}^N \varepsilon_i [f(a^i)]^{\beta k + \gamma - 1} f'(a^i) \partial_x [f(a^i)]^{\beta n + \gamma - 1} f'(a^i) \delta(x - x').$$

Expressing r.h.s. for some special set of parameters β and γ via higher moments, one can obtain exactly the Dorfman Poisson brackets investigated in [4].

If, for instance (cf. (32)),

$$A_{(m,n)}^{n-2m} = (m-n) \sum_{i=1}^N \varepsilon_i \ln \wp(a^i), \quad A_{(m,n)}^k|_{k \neq n-2m} = \frac{(m-n)^2}{k+2m-n} \sum_{i=1}^N \varepsilon_i \wp^{\frac{k+2m-n}{m-n}}(a^i), \quad m, n \in \mathbf{Z}.$$

where $\wp(z, g_2, g_3)$ is the Weiershtrass elliptic function, then corresponding Poisson bracket (25) is determined by

$$B^{ks}(\mathbf{A}) = (4k + 10m - 6n)A_{(m,n)}^{k+s+3m-2n} - (k + \frac{3m-n}{2})g_2A_{(m,n)}^{k+s+m} - (k+m)g_3A_{(m,n)}^{k+s+n}.$$

Example 6: Let us introduce the moments

$$A^k = \sum_{i=1}^N \varepsilon_i \int U^k(a^i) da^i,$$

where

$$U' = \sum_{m=0}^M \gamma_m U^m. \quad (33)$$

Then we have

$$\begin{aligned} \{A^k, A^n\} &= \sum_{i=1}^N \varepsilon_i U^k(a^i) \partial_x U^n(a^i) \delta(x-x') \\ &= \sum_{i=1}^N \varepsilon_i U^{k+n}(a^i) \delta'(x-x') + \frac{n}{k+n} \left[\sum_{i=1}^N \varepsilon_i U^{k+n}(a^i) \right]_x \delta(x-x'). \end{aligned}$$

Since (see (33))

$$\frac{1}{k+1} \sum_{i=1}^N \varepsilon_i U^{k+1}(a^i) = \sum_{m=0}^M \gamma_m A^{m+k},$$

then this Poisson bracket can be written in the most compact form

$$\{A^0, A^0\} = \sum_{n=1}^N \varepsilon_n \delta'(x-x'), \quad \{A^k, A^n\} = \sum_{m=0}^M \gamma_m [kA^{m+k+n-1} \partial_x + n \partial_x A^{m+k+n-1}] \delta(x-x').$$

These are exactly the Dorfman Poisson brackets (see [4]).

Remark: all above Poisson brackets are obtained by the phenomenological Hamiltonian approach presented in this paper. However, one can verify directly (see [30]), that, indeed, these Poisson brackets satisfy the Jacobi identity (26).

4 The integrability criteria

In this section we establish a new approach in the classification of integrable hydrodynamic chains and their integrability. If a hydrodynamic type system possesses the Hamiltonian

structure, it is not enough for integrability (the “integrability” means the existence of infinitely many conservation laws and commuting flows (see [35])). However, integrable hydrodynamic chains possessing the Hamiltonian structure can be classified. The Hamiltonian hydrodynamic reductions determine a generating function of conservation laws, which exists for integrable hydrodynamic chains only.

Theorem 2: *If a hydrodynamic chain has a reducible Poisson bracket, then this Hamiltonian hydrodynamic chain has at least one $N - 1$ parametric family of the Hamiltonian hydrodynamic reductions.*

Proof: Without loss of generality we restrict our consideration on the Kupershmidt–Manin bracket (21). Assume for simplicity, that the Hamiltonian is $\bar{\mathbf{H}}_2 = \int \mathbf{H}_2(A^0, A^1, A^2)dx$. Then the corresponding hydrodynamic chain (20) is

$$A_t^k = (kA^{k+1}\partial_x + 2\partial_x A^{k+1})\frac{\partial \mathbf{H}_2}{\partial A^2} + (kA^k\partial_x + \partial_x A^k)\frac{\partial \mathbf{H}_2}{\partial A^1} + kA^{k-1}\partial_x\frac{\partial \mathbf{H}_2}{\partial A^0}. \quad (34)$$

This hydrodynamic chain has the moment decomposition (31), where $\beta = \gamma = 1$. Then the above hydrodynamic chain reduces to the hydrodynamic type system written in the conservative form

$$a_t^i = \partial_x \left(\frac{\partial \mathbf{H}_2}{\partial A^0} + a^i \frac{\partial \mathbf{H}_2}{\partial A^1} + (a^i)^2 \frac{\partial \mathbf{H}_2}{\partial A^2} \right). \quad (35)$$

This hydrodynamic chain has at least three local conservation laws

$$\begin{aligned} A_t^0 &= \partial_x \left(A^0 \frac{\partial \mathbf{H}_2}{\partial A^1} + 2A^1 \frac{\partial \mathbf{H}_2}{\partial A^2} \right), \quad A_t^1 = \partial_x \left(A^0 \frac{\partial \mathbf{H}_2}{\partial A^0} + 2A^1 \frac{\partial \mathbf{H}_2}{\partial A^1} + 3A^2 \frac{\partial \mathbf{H}_2}{\partial A^2} - \mathbf{H}_2 \right), \\ \partial_t \mathbf{H}_2 &= \partial_x \left(2A^3 \left(\frac{\partial \mathbf{H}_2}{\partial A^2} \right)^2 + 3A^2 \frac{\partial \mathbf{H}_2}{\partial A^1} \frac{\partial \mathbf{H}_2}{\partial A^2} + A^1 \left(2 \frac{\partial \mathbf{H}_2}{\partial A^0} \frac{\partial \mathbf{H}_2}{\partial A^2} + \left(\frac{\partial \mathbf{H}_2}{\partial A^1} \right)^2 \right) + A^0 \frac{\partial \mathbf{H}_2}{\partial A^0} \frac{\partial \mathbf{H}_2}{\partial A^1} \right). \end{aligned}$$

Then the hydrodynamic reduction (35) must have the same conservation laws. From the first of them one can obtain the following restriction $\Sigma \varepsilon_m = 0$. Thus, $N - 1$ parametric family of hydrodynamic reductions is found. The above hydrodynamic reduction has the Hamiltonian form (13) while the Hamiltonian density of hydrodynamic type system \mathbf{h}_2 is the *reduced* Hamiltonian density \mathbf{H}_2 of hydrodynamic chain (34), where $A^0 = \Sigma \varepsilon_m a^m$, $A^1 = \Sigma \varepsilon_m (a^m)^2/2$, $A^2 = \Sigma \varepsilon_m (a^m)^3/3$. Then one can check the identity

$$\begin{aligned} d\mathbf{h}_2 &= \sum_{i=1}^N \frac{\partial \mathbf{h}_2}{\partial a^i} da^i = \sum_{i=1}^N \varepsilon_i \left(\frac{\partial \mathbf{H}_2}{\partial A^0} + a^i \frac{\partial \mathbf{H}_2}{\partial A^1} + (a^i)^2 \frac{\partial \mathbf{H}_2}{\partial A^2} \right) da^i \\ &= \sum_{i=1}^N \left(\frac{\partial \mathbf{H}_2}{\partial A^0} dA^0 + \frac{\partial \mathbf{H}_2}{\partial A^1} dA^1 + \frac{\partial \mathbf{H}_2}{\partial A^2} dA^2 \right) = d\mathbf{H}_2. \end{aligned}$$

Thus, indeed, $N - 1$ parametric family of hydrodynamic reductions has the Hamiltonian form, and a^k are flat coordinates.

The main statement: *if the hydrodynamic chain (34) is integrable, then above hydrodynamic reductions (35) determine the generating function of conservation laws*

$$p_t = \partial_x \left(\frac{\partial \mathbf{H}_2}{\partial A^2} p^2 + \frac{\partial \mathbf{H}_2}{\partial A^1} p + \frac{\partial \mathbf{H}_2}{\partial A^0} \right) \quad (36)$$

by the replacement $a^i \rightarrow p$. A deformation of the Riemann surface determined by the equation $\lambda = \lambda(\mathbf{A}; p)$ is described by the Gibbons equation

$$\lambda_t - \left(2 \frac{\partial \mathbf{H}_2}{\partial A^2} p + \frac{\partial \mathbf{H}_2}{\partial A^1} \right) \lambda_x = \frac{\partial \lambda}{\partial p} \left[p_t - \partial_x \left(\frac{\partial \mathbf{H}_2}{\partial A^2} p^2 + \frac{\partial \mathbf{H}_2}{\partial A^1} p + \frac{\partial \mathbf{H}_2}{\partial A^0} \right) \right].$$

Thus, the problem of the classification of integrable Hamiltonian hydrodynamic chains is reduced to the problem of the classification of generating functions of conservation laws (see the section “General case” in [32]).

Here we enumerate **four** different tools allowing to prove an integrability of such hydrodynamic chains.

1. Extra conservation law and method of pseudopotentials. For instance, suppose the hydrodynamic chain (34) is integrable, then the first commuting flow determined by the Hamiltonian density $\mathbf{H}_3(A^0, A^1, A^2, A^3)$ creates the corresponding generating function of conservation laws is

$$p_{t^2} = \partial_x \left(\frac{\partial \mathbf{H}_3}{\partial A^3} p^3 + \frac{\partial \mathbf{H}_3}{\partial A^2} p^2 + \frac{\partial \mathbf{H}_3}{\partial A^1} p + \frac{\partial \mathbf{H}_3}{\partial A^0} \right). \quad (37)$$

The compatibility condition $\partial_{t^2}(p_t) = \partial_t(p_{t^2})$ leads to exactly the same system of nonlinear PDE’s in involution as obtained by the tensor approach in [10]. The equations (36) and (37) are nothing but pseudopotentials for corresponding 2+1 quasilinear equation (see, for instance, (4) – the compatibility condition $\partial_{t^2}(p_t) = \partial_t(p_{t^2})$ yields the Khokhlov–Zabolotzkaya equation, see (2) and below). Thus, we *extend* the concept of pseudopotentials on the theory of integrable hydrodynamic chains (see the section 5).

2. Reciprocal transformations. The Hamiltonian density \mathbf{H}_2 can be *linear* $\mathbf{H}_2 = A^2 + f(A^0, A^1)$, *quasilinear* $\mathbf{H}_2 = g(A^0, A^1)A^2 + f(A^0, A^1)$ and *fully nonlinear* $\mathbf{H}_2(A^0, A^1, A^2)$. Thus, the generating functions (36) can be separated on three sub-classes

$$\begin{aligned} p_t &= \partial_x \left(p^2 + \frac{\partial f(A^0, A^1)}{\partial A^1} p + \frac{\partial f(A^0, A^1)}{\partial A^0} \right), \\ p_t &= \partial_x \left[g(A^0, A^1) p^2 + \left(\frac{\partial g(A^0, A^1)}{\partial A^1} p + \frac{\partial g(A^0, A^1)}{\partial A^0} \right) A^2 + \frac{\partial f(A^0, A^1)}{\partial A^1} p + \frac{\partial f(A^0, A^1)}{\partial A^0} \right], \\ p_t &= \partial_x \left(\frac{\partial \mathbf{H}_2}{\partial A^2} p^2 + \frac{\partial \mathbf{H}_2}{\partial A^1} p + \frac{\partial \mathbf{H}_2}{\partial A^0} \right). \end{aligned}$$

The difference between these sub-classes is obvious. The **main claim** is that the factors $g(A^0, A^1)$ and $\partial \mathbf{H}_2 / \partial A^2$ can be removed under some special reciprocal transformations. Thus, an integrability of an arbitrary (*quasilinear* or *fully nonlinear*) Hamiltonian integrable hydrodynamic chain is reduced to an integrability of a *linear* Hamiltonian integrable hydrodynamic chain (see the sections 6 and 7).

3. Asymptotic at the vicinity of each singular point. Suppose we have some generating function of conservation laws of polynomial type (see, for instance, the above *linear* case)

$$p_t = \partial_x \left(\frac{p^N}{N} + a_0 p^{N-1} + a_1 p^{N-2} + \dots + a_{N-1} \right).$$

Then the generation function of conservation law densities p can be determined by the series

$$p = \lambda + \mathbf{H}_0 + \frac{\mathbf{H}_1}{\lambda} + \frac{\mathbf{H}_2}{\lambda^2} + \frac{\mathbf{H}_3}{\lambda^3} + \dots \quad (38)$$

If a hydrodynamic chain is integrable, then such an expansion exists (see the section 7).

4. *Hydrodynamic reductions.* This is the most universal approach (see [7], [16], [32]). Let us consider the equation of the Riemann surface $\lambda = \lambda(\mathbf{A}; p)$ satisfying the Gibbons–Tsarev equation (cf. (8); see [32])

$$\lambda_t - \frac{\partial \psi}{\partial p} \lambda_x = \frac{\partial \lambda}{\partial p} [p_t - \partial_x \psi(\mathbf{A}; p)].$$

If $\lambda = \text{const}$, then the Gibbons equation reduces to the generating function of conservation laws $p_t = \partial_x \psi(\mathbf{A}; p)$. Suppose the moments A^n are functions of N Riemann invariants r^k ; i.e. we seek N component hydrodynamic reductions written in the diagonal form

$$r_t^i = \mu^i(\mathbf{r}) r_x^i,$$

where $\mu^i = \partial \psi / \partial p|_{p=p^i}$, $r^i = \lambda|_{p=p^i}$ and N *distinct* values p^k are determined by the algebraic equation $\partial \lambda / \partial p = 0$. Then a consistency of the above hydrodynamic type system and its generating function of conservation laws yields the extended Gibbons–Tsarev system (see [32]) describing all admissible functions $\psi(\mathbf{A}; p)$ (see details below). This approach is adopted for the special class of the hydrodynamic reductions

$$a_t^i = \partial_x \psi(\mathbf{A}; a^i),$$

which has the Hamiltonian form (19). In this case all moments A^n can be explicitly expressed via flat coordinates a^k . The method of N component hydrodynamic reductions (see (5) in the Riemann invariants r^k and (10) in the conservation law densities a^i) leads to the Gibbons–Tsarev system, whose general solution is parameterized by N arbitrary functions of a single variable. The method of N component *Hamiltonian* hydrodynamic reductions **reduces the Gibbons–Tsarev system to the sole ODE** (see for instance (14)). Let us consider, for instance, (10). If the hydrodynamic reduction (9) has the Hamiltonian structure (13), then the moment A^0 is a some function of $\Delta = \Sigma \varepsilon_n a^n$. Thus, the system of nonlinear PDE’s (10) reduces to the ODE $(A^0)'' = 0$. Thus, indeed $A^0 = \Sigma \varepsilon_n a^n$ in accordance with (14).

Thus, the Hamiltonian approach consists of following steps:

1. A “moment decomposition” search for given Poisson bracket (see the previous section).
2. A transformation to the *linear* Hamiltonian case.
3. An asymptotic investigation of generating functions of conservation laws.
4. A derivation of the *extended* Gibbons–Tsarev system in involution describing “integrable” Hamiltonians and N component hydrodynamic reductions parameterized by N arbitrary functions of a single variable (see [32]).

5 Classification of simplest Hamiltonian hydrodynamic chains

The Kupershmidt Poisson brackets

$$\{A^k, A^n\} = [(k + \gamma)A^{k+n-M}\partial_x + (n + \gamma)\partial_x A^{k+n-M}]\delta(x - x'), \quad (39)$$

where M is an arbitrary integer, is a particular case of so-called M -brackets (see [30])

$$\{A^k, A^n\} = [B^{k,n}\partial_x + \partial_x B^{n,k}]\delta(x - x'),$$

where the coefficients $B^{k,n}$ depend on the first $k + n - M$ elements only (if $k + n < M$, then corresponding coefficients are constants). These Poisson brackets are equivalent to each other under the re-numeration $A^k \rightarrow A^{k+s}$ (where s is an appropriate integer). The case $M = 1$ was considered in [22], the case $M = 0$ was considered in [21], the case $M = -1$ was considered in [23].

In this section the classification of integrable hydrodynamic chains determined by the Kupershmidt Poisson brackets (see (39), $M = -1$ and [23], $\gamma \neq 0$)

$$C_{t^1}^k = [(k + \gamma)C^{k+n+1}\partial_x + (n + \gamma)\partial_x C^{k+n+1}]\frac{\delta\bar{\mathbf{H}}_0}{\delta C^n}, \quad k = 0, 1, 2, \dots,$$

where $\bar{\mathbf{H}}_0 = \int \mathbf{H}_0(C^0)dx$, is given. If the Hamiltonian density $\mathbf{H}_0(C^0)$ is an *arbitrary* function, then the hydrodynamic chain

$$C_{t^1}^k = \gamma \mathbf{H}'_0(C^0)C_x^{k+1} + (k + 2\gamma)C^{k+1}\mathbf{H}''_0(C^0)C_x^0, \quad k = 0, 1, 2, \dots \quad (40)$$

has just the conservation law of the energy

$$\partial_{t^1}\mathbf{H}_0(C^0) = \partial_x[\gamma\mathbf{H}_0'^2(C^0)C^1].$$

5.1 The first integrability criterion (extra conservation law)

Conjecture 3: If a hydrodynamic chain has an *extra* conservation law, then this hydrodynamic chain is integrable, i.e. this hydrodynamic chain has infinitely many conservation laws and commuting flows (see examples in [21]).

Indeed, a second conservation law can be found in the form

$$\partial_{t^1}[f(C^0)C^1] = \partial_x \left[\frac{\gamma}{2}f'(C^0)\mathbf{H}'_0(C^0)(C^1)^2 + \gamma f(C^0)\mathbf{H}'_0(C^0)C^2 \right],$$

where

$$3f'\mathbf{H}_0'' = f''\mathbf{H}'_0, \quad (\gamma + 1)f\mathbf{H}_0'' = \gamma f'\mathbf{H}'_0. \quad (41)$$

Thus, the integrable hydrodynamic chain (see [23])

$$C_{t^1}^k = \frac{\gamma(1 - \gamma)}{(1 - 2\gamma)^2}(C^0)^{\frac{3\gamma-1}{1-2\gamma}}[(1 - 2\gamma)C^0C_x^{k+1} + (k + 2\gamma)C^{k+1}C_x^0], \quad k = 0, 1, 2, \dots \quad (42)$$

is determined by the Hamiltonian density $\mathbf{H}_0 = (C^0)^{\frac{1-\gamma}{1-2\gamma}}$ in general case ($\gamma \neq -1$, $\gamma \neq 1$ and $\gamma \neq 1/2$). Then the second conservation law density $\mathbf{H}_1 = C^1(C^0)^{\frac{1+\gamma}{1-2\gamma}}$ determines the first commuting flow

$$C_{t^2}^k = (1+\gamma)(C^0)^{\frac{1+\gamma}{1-2\gamma}}C_x^{k+2} + \frac{\gamma(1+\gamma)}{1-2\gamma}C^1(C^0)^{\frac{3\gamma}{1-2\gamma}}C_x^{k+1} + \frac{(1+\gamma)(k+2\gamma)}{1-2\gamma}C^{k+1}(C^0)^{\frac{3\gamma}{1-2\gamma}}C_x^1 \\ + \frac{1+\gamma}{1-2\gamma}(C^0)^{\frac{5\beta-1}{1-2\beta}}[(k+1+2\gamma)C^0C^{k+2} + \frac{3\gamma(k+2\gamma)}{1-2\gamma}C^1C^{k+1}]C_x^0, \quad k = 0, 1, 2, \dots$$

Remark: These hydrodynamic chains (for the **integer** values $M = (1-3\gamma)/(1-2\gamma)$) were derived in [2] (see also [26]) via the Lax formulation. They are first members of so-called dispersionless limit of the M -dDym hierarchy. In this paper we recover the Hamiltonian structure for these hydrodynamic chains. We believe that the Lax formulation is connected with this Hamiltonian structure. However, this is open question.

In particular cases:

if $\gamma = 1$, then $(\mathbf{H}_0 = \ln C^0, \mathbf{H}_1 = C^1(C^0)^{-2})$

$$C_{t^1}^k = \frac{1}{C^0}C_x^{k+1} - (k+2)C^{k+1}\frac{C_x^0}{(C^0)^2}, \quad k = 0, 1, 2, \dots, \quad (43)$$

$$C_{t^2}^k = 2(C^0)^{-2}C_x^{k+2} - 2C^1(C^0)^{-3}C_x^{k+1} - 2(k+2)C^{k+1}(C^0)^{-3}C_x^1$$

$$-2(C^0)^{-4}[(k+3)C^0C^{k+2} - 3(k+2)C^1C^{k+1}]C_x^0, \quad k = 0, 1, 2, \dots;$$

if $\gamma = 1/2$, then $(\mathbf{H}_0 = e^{C^0}, \mathbf{H}_1 = C^1e^{3C^0})$

$$C_{t^1}^k = \frac{1}{2}e^{C^0}C_x^{k+1} + (k+1)C^{k+1}e^{C^0}C_x^0, \quad k = 0, 1, 2, \dots, \quad (44)$$

$$C_{t^2}^k = \frac{3}{2}e^{3C^0}C_x^{k+2} + \frac{3}{2}C^1e^{3C^0}C_x^{k+1} + 3(k+1)C^{k+1}e^{3C^0}C_x^1$$

$$+ 3e^{3C^0}[(k+2)C^{k+2} + 3(k+1)C^{k+1}C^1]C_x^0, \quad k = 0, 1, 2, \dots;$$

if $\gamma = -1$, then the second conservation law

$$C_{t^1}^1 = -\partial_x[C^2\mathbf{H}'_0(C^0)].$$

exists for **any** Hamiltonian density $\mathbf{H}_0(C^0)$. Thus, in this case the hydrodynamic chain

$$C_{t^1}^k = -\mathbf{H}'_0(C^0)C_x^{k+1} + (k-2)C^{k+1}\mathbf{H}''_0(C^0)C_x^0, \quad k = 0, 1, 2, \dots$$

is integrable, if the **third** conservation law

$$\partial_{t^1}\mathbf{H}_2(C^0, C^1, C^2) = \partial_x\mathbf{G}_2(C^0, C^1, C^2, C^3)$$

exists. Then,

$$\mathbf{H}_2 = -\frac{3}{2}(C^0)^{1/3}C^2 - \frac{1}{4}(C^0)^{-2/3}(C^1)^2,$$

$$\mathbf{G}_2 = C^3 + \frac{C^1C^2}{3C^0} + \frac{(C^1)^3}{27(C^0)^2}$$

and the integrable hydrodynamic chain is

$$C_{t^1}^k = -\frac{2}{9}(C^0)^{-4/3}[3C^0C_x^{k+1} + (k-2)C^{k+1}C_x^0], \quad k = 0, 1, 2, \dots,$$

where the Hamiltonian density is $\mathbf{H}_0 = (C^0)^{2/3}$.

5.2 The second integrability criterion (generating functions of conservation laws)

Hydrodynamic reductions of the Hamiltonian chain (42) one can seek with the aid of the moment decomposition given in the form (31)

$$C^k = \frac{\gamma - 1}{k + 2\gamma - 1} \sum_{i=1}^N \varepsilon_i (c^i)^{\frac{k+2\gamma-1}{\gamma-1}}, \quad (45)$$

if $\gamma \neq (1 - K)/2$, where $K = 0, 1, 2, \dots$

Then the corresponding hydrodynamic type system

$$c_{t^1}^i = (\gamma - 1)\partial_x[(c^i)^{\frac{\gamma}{\gamma-1}}\mathbf{h}'_0(C^0)] \quad (46)$$

has the local Hamiltonian structure (13)

$$c_{t^1}^i = \partial_x \left[\frac{\gamma - 1}{\varepsilon_i} \frac{\delta \bar{\mathbf{h}}_0}{\delta c^i} \right], \quad (47)$$

where the Hamiltonian $\bar{\mathbf{h}}_0 = \int \mathbf{h}_0(C^0)dx$.

Remark: If $\gamma = (1 - K)/2$, then

$$C^k|_{k \neq K} = \frac{1 + K}{2(K - k)} \sum_{i=1}^N \varepsilon_i (c^i)^{\frac{2(K-k)}{1+K}}, \quad C^K = \sum_{i=1}^N \varepsilon_i \ln c^i.$$

For instance, if $\gamma = 1/2$ (see (44)), then

$$C^0 = \sum_{i=1}^N \varepsilon_i \ln c^i, \quad C^k = -\frac{1}{2k} \sum_{i=1}^N \varepsilon_i (c^i)^{-2k}, \quad k = 1, 2, \dots$$

In such cases the compatibility of the above moment decomposition (45) with the hydrodynamic chains (42) and (44) brings to the extra parametric restriction (cf. the previous sections)

$$\sum_{i=1}^N \varepsilon_i = 0. \quad (48)$$

If $\gamma = 1$, then hydrodynamic reductions of the Hamiltonian chain (43)

$$C_{t^1}^k = \mathbf{H}'_0(C^0)C_x^{k+1} + (k+2)C^{k+1}\mathbf{H}''_0(C^0)C_x^0, \quad k = 0, 1, 2, \dots$$

one can seek with the aid of the moment decomposition in the *degenerate* form (31)

$$C^k = \frac{1}{k+1} \sum_{i=1}^N \varepsilon_i e^{(k+1)c^i}.$$

Then the corresponding hydrodynamic type system

$$c_{t^1}^i = \partial_x \left[\frac{e^{c^i}}{C^0} \right] \quad (49)$$

has the local Hamiltonian structure (cf. (13))

$$c_{t^1}^i = \partial_x \left[\frac{1}{\varepsilon_i} \frac{\delta \bar{\mathbf{h}}_0}{\delta c^i} \right],$$

where the Hamiltonian $\bar{\mathbf{h}}_0 = \int \ln C^0 dx$.

I **postulate** the existence of the generating function of conservation laws

$$s_{t^1} = (\gamma - 1) \partial_x [s^{\frac{\gamma}{\gamma-1}} \mathbf{H}'_0(C^0)], \quad \gamma \neq 1; \quad s_{t^1} = \partial_x \left(\frac{e^s}{C^0} \right), \quad \gamma = 1 \quad (50)$$

obtained from the above hydrodynamic reductions (46) and (49) by the formal replacement $c^i \rightarrow s$. This is the **main ansatz** of this Hamiltonian approach (see (36)).

In the same way the generating function of conservation laws for the first commuting flow can be found

$$s_{t^2} = (\gamma - 1) \partial_x \left(s^{\frac{\gamma+1}{\gamma-1}} \frac{\partial \mathbf{H}_1}{\partial C^1} + s^{\frac{\gamma}{\gamma-1}} \frac{\partial \mathbf{H}_1}{\partial C^0} \right), \quad \gamma \neq 1; \quad s_{t^2} = \partial_x \left(\frac{e^{2s}}{(C^0)^2} - 2 \frac{e^s C^1}{(C^0)^3} \right), \quad \gamma = 1.$$

The substitution 2+1 quasilinear system

$$\begin{aligned} C_{t^1}^0 &= 2\gamma C^1 \left(\frac{\partial \mathbf{H}_0}{\partial C^0} \right)_x + \gamma \frac{\partial \mathbf{H}_0}{\partial C^0} C_x^1, & C_{t^1}^1 &= (2\gamma + 1) C^2 \left(\frac{\partial \mathbf{H}_0}{\partial C^0} \right)_x + \gamma \frac{\partial \mathbf{H}_0}{\partial C^0} C_x^2, \\ C_{t^2}^0 &= 2\gamma C^1 \left(\frac{\partial \mathbf{H}_1}{\partial C^0} \right)_x + \gamma \frac{\partial \mathbf{H}_1}{\partial C^0} C_x^1 + (2\gamma + 1) C^2 \left(\frac{\partial \mathbf{H}_1}{\partial C^1} \right)_x + (\gamma + 1) \frac{\partial \mathbf{H}_1}{\partial C^1} C_x^2 \end{aligned}$$

in the compatibility condition $(s_{t^1})_{t^2} = (s_{t^2})_{t^1}$ yields the couple of equations

$$\begin{aligned} (\gamma + 1) \frac{\partial \mathbf{H}_1}{\partial C^1} \frac{\partial^2 \mathbf{H}_0}{\partial (C^0)^2} &= \gamma \frac{\partial \mathbf{H}_0}{\partial C^0} \frac{\partial^2 \mathbf{H}_1}{\partial C^0 \partial C^1}, & \frac{\partial^2 \mathbf{H}_1}{\partial (C^1)^2} &= 0, \\ \frac{\partial \mathbf{H}_0}{\partial C^0} \frac{\partial^2 \mathbf{H}_1}{\partial (C^0)^2} &= \frac{\partial \mathbf{H}_1}{\partial C^0} \frac{\partial^2 \mathbf{H}_0}{\partial (C^0)^2} + 2C^1 \frac{\partial^2 \mathbf{H}_0}{\partial (C^0)^2} \frac{\partial^2 \mathbf{H}_1}{\partial C^0 \partial C^1}, \end{aligned}$$

whose solutions are the *same* as solutions of the system (41).

5.3 The third integrability criterion (the Hamiltonian hydrodynamic reductions)

Suppose the Hamiltonian hydrodynamic chain (40) is integrable. Then its Hamiltonian hydrodynamic reduction (47) must be consistent with the generating function of conservation laws (50), which is a particular case of more general generating function of conservation laws $s_t = \partial_x(V(s)v)$ considered in details in [32]. The compatibility condition $\partial_i(\partial_k s) = \partial_k(\partial_i s)$ (written via flat coordinates c^k ; see (45)) yields the ODE

$$\gamma \mathbf{h}'_0 \mathbf{h}'''_0 = (3\gamma - 1) \mathbf{h}''_0{}^2,$$

whose solution $\mathbf{h}_0 = (C^0)^{\frac{1-\gamma}{1-2\gamma}}$ again (see (41)), where

$$\partial_i s = \frac{\varepsilon_i(c^i)^{\frac{\gamma}{\gamma-1}} s^{\frac{\gamma}{\gamma-1}}}{(c^i)^{\frac{1}{\gamma-1}} - s^{\frac{1}{\gamma-1}}} \left(\frac{\gamma}{\gamma-1} \frac{\mathbf{h}'_0}{\mathbf{h}''_0} + \sum \frac{\varepsilon_k(c^k)^{\frac{2\gamma}{\gamma-1}}}{(c^k)^{\frac{1}{\gamma-1}} - s^{\frac{1}{\gamma-1}}} \right)^{-1}.$$

Since $\partial_i s = -\partial_i \lambda / \partial_s \lambda$ (see [31]), the equation of the Riemann surface $\lambda(\mathbf{c}; s)$ can be found in quadratures

$$d\lambda = s^{-3} \sum \frac{\varepsilon_k(c^k)^{\frac{2\gamma-1}{\gamma-1}}}{(c^k)^{\frac{1}{\gamma-1}} - s^{\frac{1}{\gamma-1}}} ds - s^{-2} \sum \frac{\varepsilon_k(c^k)^{\frac{\gamma}{\gamma-1}}}{(c^k)^{\frac{1}{\gamma-1}} - s^{\frac{1}{\gamma-1}}} dc^k.$$

Taking into account the moment decomposition (45), the equation of the Riemann surface can be written in the form

$$\lambda = \sum_{k=0}^{\infty} C^k s^{\frac{1-2\gamma-k}{\gamma-1}}. \quad (51)$$

Thus, this equation of the Riemann mapping is valid for whole hydrodynamic chain (as well as for *any* its hydrodynamic reduction). Indeed,

Theorem 3: *The Gibbons equation*

$$\lambda_{t^1} - \frac{\gamma(1-\gamma)}{1-2\gamma} s^{\frac{1}{\gamma-1}} (C^0)^{\frac{\gamma}{1-2\gamma}} \lambda_x = \frac{\partial \lambda}{\partial s} \left[s_{t^1} + \frac{(1-\gamma)^2}{1-2\gamma} \partial_x (s^{\frac{\gamma}{\gamma-1}} (C^0)^{\frac{\gamma}{1-2\gamma}}) \right] \quad (52)$$

connects the Hamiltonian hydrodynamic chain (42) with the Riemann mapping (51) ($\gamma \neq 1, \gamma \neq 1/2$).

Proof: can be obtained by the direct substitution (42) and (51) in (52).

Remark: The Gibbons equation

$$\lambda_{t^1} - \frac{e^s}{C^0} \lambda_x = \frac{\partial \lambda}{\partial s} \left[s_{t^1} - \partial_x \left(\frac{e^s}{C^0} \right) \right]$$

connects the Hamiltonian hydrodynamic chain (43) with the Riemann mapping

$$\lambda = \sum_{k=0}^{\infty} C^k e^{-(k+1)s}.$$

The Gibbons equation

$$\lambda_{t^1} - \frac{e^{C^0}}{2s^2} \lambda_x = \frac{\partial \lambda}{\partial s} \left[s_{t^1} + \partial_x \left(\frac{e^{C^0}}{2s} \right) \right]$$

connects the Hamiltonian hydrodynamic chain (44) with the Riemann mapping

$$\lambda = -\ln s + \sum_{k=0}^{\infty} C^k s^{2k}.$$

Let us consider two hydrodynamic chains

$$C_t^k = \sum_{n=0}^{k+1} F_n^k(\mathbf{C}) C_x^n, \quad \tilde{C}_t^k = \sum_{n=0}^{k+1} \tilde{F}_n^k(\tilde{\mathbf{C}}) \tilde{C}_x^n,$$

related by the infinitely many *invertible* transformations

$$\tilde{C}^k = \tilde{C}^k(C^0, C^1, \dots, C^k), \quad k = 0, 1, 2, \dots \quad (53)$$

Theorem 4: *The hydrodynamic chain*

$$\partial_t C_{(\alpha)}^k = \frac{\gamma}{\alpha} (C_{(\alpha)}^0)^{\gamma/\alpha-1} [\alpha C_{(\alpha)}^0 \partial_x C_{(\alpha)}^{k+1} + (k+1-\alpha) C_{(\alpha)}^{k+1} \partial_x C_{(\alpha)}^0] \quad (54)$$

is equivalent to the Hamiltonian hydrodynamic chain (42)

$$C_t^k = \frac{\gamma}{1-2\gamma} (C^0)^{\frac{3\gamma-1}{1-2\gamma}} [(1-2\gamma) C^0 C_x^{k+1} + (k+2\gamma) C^{k+1} C_x^0],$$

where $C^k \equiv C_{(1-2\gamma)}^k$.

Proof: The hydrodynamic chain (54) satisfies the Gibbons equation (52)

$$\lambda_t - \gamma s^{\frac{1}{\gamma-1}} (C^0)^{\gamma/\alpha} \lambda_x = \frac{\partial \lambda}{\partial s} [s_t + (\gamma-1) \partial_x (s^{\frac{\gamma}{\gamma-1}} (C^0)^{\gamma/\alpha})],$$

where $s = p^{\gamma-1}$ and the Riemann mapping (cf. (51)) is given by

$$\lambda = \sum_{k=0}^{\infty} C_{(\alpha)}^k p^{\alpha-k} \equiv \left[\sum_{k=0}^{\infty} C^k p^{1-2\gamma-k} \right]^{\frac{\alpha}{1-2\gamma}}.$$

Thus, the invertible transformations (53) can be derived from the above formula.

Remark: The hydrodynamic chain (40) is a particular case of three parametric hydrodynamic chain (see [31])

$$C_t^k = \mathbf{H}'_0(C^0) C_x^{k+1} + [(\alpha k + 2) C^{k+1} + \beta k C^k + \gamma k C^{k-1}] \partial_x \mathbf{H}'_0(C^0),$$

determined by the Poisson bracket

$$\{C^k, C^n\} = [B_{k,n} \partial_x + \partial_x B_{n,k}] \delta(x - x'),$$

where

$$B_{k,n} = (\alpha k + 1)A^{k+n+1} + \beta k A^{k+n} + \gamma k A^{k+n-1}.$$

This Poisson bracket looks more complicated than (39). The moment decomposition

$$C^k = \sum \int V_i'^k dz_i,$$

where the $V(s)$ satisfies the ODE

$$2VV'' = \alpha V'^2 + \beta V' + \gamma$$

is more complicated than (45). However, a complexity of the generating function of conservation laws (see [31]) is the same as in the above example (50)

$$s_t = \partial_x [\mathbf{h}'(\Delta) z'(s)],$$

where the Hamiltonian density can be of three types $\mathbf{h}(\Delta) = \exp \Delta$, $\mathbf{h}(\Delta) = \ln \Delta$, $\mathbf{h}(\Delta) = \Delta^\varepsilon$ and function $z(s)$ is given in the implicit form

$$s = \int \frac{dz}{\sqrt{2V(z)}}.$$

6 Hydrodynamic chains associated with the Kupershmidt brackets

The method of hydrodynamic reductions was established in [16] (for a description of solutions of the Benney hydrodynamic chain) and developed in [7] (see also [8]) for a classification of 2+1 quasilinear systems. In this paper this method is applied to classification of integrable hydrodynamic chains (see [21]). The second example is given by the Hamiltonian hydrodynamic chain

$$B_{t^1}^k = (1+\beta) \frac{\partial \mathbf{H}_1}{\partial B^1} B_x^{k+1} + \beta \frac{\partial \mathbf{H}_1}{\partial B^0} B_x^k + (k+1+2\beta) B^{k+1} \left(\frac{\partial \mathbf{H}_1}{\partial B^1} \right)_x + (k+2\beta) B^k \left(\frac{\partial \mathbf{H}_1}{\partial B^0} \right)_x, \quad k = 0, 1, \dots$$

associated with the Kupershmidt Poisson bracket

$$\{B^k, B^n\} = [(k+\beta)B^{k+n}\partial_x + (n+\beta)\partial_x B^{k+n}]\delta(x-x'). \quad (55)$$

If the Hamiltonian density \mathbf{H}_1 is an arbitrary, then this hydrodynamic chain has two conservation laws only. The conservation law of the momentum

$$B_{t^1}^0 = \partial_x [(1+2\beta)B^1 \frac{\partial \mathbf{H}_1}{\partial B^1} + 2\beta B^0 \frac{\partial \mathbf{H}_1}{\partial B^0} - \beta \mathbf{H}_1] \quad (56)$$

and the conservation law of the energy

$$\partial_{t^1} \mathbf{H}_1 = \partial_x \left[(1+\beta)B^2 \left(\frac{\partial \mathbf{H}_1}{\partial B^1} \right)^2 + (1+2\beta)B^1 \frac{\partial \mathbf{H}_1}{\partial B^0} \frac{\partial \mathbf{H}_1}{\partial B^1} + \beta B^0 \left(\frac{\partial \mathbf{H}_1}{\partial B^0} \right)^2 \right].$$

Taking into account that this Poisson bracket is reducible by the moment decomposition (31) (where $\beta \rightarrow 1/\beta$, $\gamma = 2$)

$$B^k = \frac{\beta}{k + 2\beta} \sum_{i=1}^N \varepsilon_i (b^i)^{k/\beta+2}, \quad (57)$$

the Hamiltonian hydrodynamic chain (see (55))

$$B_{tm}^k = [(k + \beta)B^{k+n}\partial_x + (n + \beta)\partial_x B^{k+n}] \frac{\delta \bar{\mathbf{H}}_m}{\delta B^n}, \quad k = 0, 1, 2, \dots \quad (58)$$

reduces to the hydrodynamic type system

$$b_{tm}^i = \beta \partial_x \left((b^i)^{1+n/\beta} \frac{\partial \mathbf{h}_m}{\partial B^n} \right) \quad (59)$$

where $\bar{\mathbf{H}}_m = \int \mathbf{H}_m(B^0, B^1, \dots, B^m) dx$ and $\mathbf{h}_m(\mathbf{b}) = \mathbf{H}_m|_{B^k=B^k(\mathbf{b})}$. Thus, a description of all admissible Hamiltonian densities \mathbf{H}_m can be derived from the consistency of the above hydrodynamic type system and the generating function of conservation laws

$$q_{tm} = \beta \partial_x \left(q^{1+n/\beta} \frac{\partial \mathbf{h}_m}{\partial B^n} \right). \quad (60)$$

Then the above hydrodynamic type system (59) in the Riemann invariants has the form

$$r_{tm}^i = (n + \beta)(q^i)^{n/\beta} \frac{\partial \mathbf{h}_m}{\partial B^n} r_x^i.$$

In general case the Gibbons–Tsarev system can be derived from the compatibility conditions $\partial_{ti}(\partial_{tj}q) = \partial_{tj}(\partial_{ti}q)$, where $\partial_i \equiv \partial/\partial r^i$ (cf. (6))

$$\partial_i q = \frac{\beta q^{1+n/\beta} \partial_i \left(\frac{\partial \mathbf{h}_m}{\partial B^n} \right)}{(k + \beta)[(q^i)^{k/\beta} - q^{k/\beta}] \frac{\partial \mathbf{h}_m}{\partial B^k}},$$

and from the compatibility conditions $\partial_i(\partial_k B^n) = \partial_k(\partial_i B^n)$, where $i \neq k$ and $n = 1, 2, \dots, m$.

Without loss of generality let us restrict our consideration on the Hamiltonian density $\mathbf{H}_1(B^0, B^1)$. Then

$$\partial_i q = \frac{\beta}{\beta + 1} \frac{q}{\partial \mathbf{h}_1 / \partial B^1} \frac{\partial_i(\partial \mathbf{h}_1 / \partial B^0) + q^{1/\beta} \partial_i(\partial \mathbf{h}_1 / \partial B^1)}{(q^i)^{1/\beta} - q^{1/\beta}}.$$

The last step *before* a computation of the compatibility conditions $\partial_i(\partial_k q) = \partial_k(\partial_i q)$ is the substitution the link (into the above expression for each $\partial_i q$)

$$\partial_i B^1 = \frac{(\beta + 1)(q^i)^{1/\beta} \frac{\partial \mathbf{h}_1}{\partial B^1} - (2\beta + 1)B^1 \frac{\partial^2 \mathbf{h}_1}{\partial B^0 \partial B^1} - 2\beta B^0 \frac{\partial^2 \mathbf{h}_1}{\partial (B^0)^2}}{(\beta + 1) \frac{\partial \mathbf{h}_1}{\partial B^1} + (2\beta + 1)B^1 \frac{\partial^2 \mathbf{h}_1}{\partial (B^1)^2} + 2\beta B^0 \frac{\partial^2 \mathbf{h}_1}{\partial B^0 \partial B^1}} \partial_i B^0,$$

which is a consequence on the first equation of the hydrodynamic chain (see (56)).

The compatibility conditions $\partial_i(\partial_k B^1) = \partial_k(\partial_i B^1)$ imply a last restriction on the dependence of the Hamiltonian density \mathbf{h}_1 on B^0 and B^1 .

The same Gibbons–Tsarev system but written via field variables b^k (see (59)) can be derived from the compatibility conditions $\partial_i(\partial_j q) = \partial_j(\partial_i q)$, where $\partial_i \equiv \partial/\partial b^i$ and

$$\partial_i q = \frac{\beta}{\beta+1} \frac{(\sum (b^k)^{1+1/\beta} \partial_k q - q^{1+1/\beta}) \partial_i(\partial \mathbf{h}_1 / \partial B^1) + (\sum b^k \partial_k q - q) \partial_i(\partial \mathbf{h}_1 / \partial B^0)}{[q^{1/\beta} - (b^i)^{1/\beta}] \partial \mathbf{h}_1 / \partial B^0}.$$

Before a substitution of the above expression in the compatibility conditions $\partial_i(\partial_k q) = \partial_k(\partial_i q)$, one must compute $\Sigma(b^k)^{1+1/\beta} \partial_k q$ and $\Sigma b^k \partial_k q$. This investigation in details will be presented elsewhere.

This is a particular case of more general problem: a description of all admissible functions $\psi(B^0, B^1; q)$ (i.e. the classification of integrable hydrodynamic chains) determined by the generating function of conservation laws $q_t = \partial_x \psi(B^0, B^1; q)$ (see details in [32]). However, if the compatibility conditions $\partial_i(\partial_k q) = \partial_k(\partial_i q)$ imply the extended Gibbons–Tsarev system describing all admissible functions $\psi(B^0, B^1; q)$ and corresponding N component reductions parameterized by N arbitrary functions of a single variable, if the compatibility conditions $\partial_i(\partial_k q) = \partial_k(\partial_i q)$ for integrable 2+1 quasilinear systems like dKP (see [16] and [7]) imply the Gibbons–Tsarev system describing N component reductions parameterized by N arbitrary functions of a single variable only, the compatibility conditions $\partial_i(\partial_k q) = \partial_k(\partial_i q)$ for integrable hydrodynamic chains connected with the Kupershmidt Poisson bracket *reduce the Gibbons–Tsarev system to the sole ODE* only. It means that the advantage of this Hamiltonian hydrodynamic reductions approach in the application to the Hamiltonian hydrodynamic chains, whose Poisson brackets are reducible, is that the Gibbons–Tsarev system implies just a restriction on the Hamiltonian density.

Integrable hydrodynamic chains associated with the Kupershmidt Poisson bracket (55) can be determined by the *linear* Hamiltonian density $\mathbf{H}_1 = B^1 + f(B^0)$, the *quasilinear* Hamiltonian density $\mathbf{H}_1 = g(B^0)B^1 + f(B^0)$ and the *fully nonlinear* Hamiltonian density $\mathbf{H}_1(B^0, B^1)$.

6.1 The *linear* Hamiltonian density

The integrability of the hydrodynamic chain determined by the *linear* Hamiltonian density $\mathbf{H}_1 = B^1 + f(B^0)$

$$B_{t^1}^k = B_x^{k+1} + \frac{\beta}{\beta+1} f'(B^0) B_x^k + \frac{k+2\beta}{1+\beta} B^k f''(B^0) B_x^0, \quad k = 0, 1, \dots$$

can be examined with the aid of the method of hydrodynamic reductions, where the generating function of conservation laws is given by (see (60))

$$q_{t^1} = \beta \partial_x (q^{1+1/\beta} + f'(B^0) q). \quad (61)$$

In this case the Gibbons–Tsarev system can be derived from the compatibility conditions $\partial_i(\partial_k q) = \partial_k(\partial_i q)$ and $\partial_i(\partial_k B^1) = \partial_k(\partial_i B^1)$, where $\partial_k \equiv \partial/\partial r^k$ and

$$\partial_i q = \frac{\beta}{\beta+1} q \frac{f''(B^0) \partial_i B^0}{(q^i)^{1/\beta} - q^{1/\beta}}, \quad \partial_i B^1 = [(q^i)^{1/\beta} - \frac{2\beta}{\beta+1} B^0 f''(B^0)] \partial_i B^0.$$

The function $f(B^0)$ also can be found from the compatibility conditions $\partial_i(\partial_k q) = \partial_k(\partial_i q)$, where $\partial_k \equiv \partial/\partial b^k$ and

$$\partial_i q = \frac{q}{q^{1/\beta} - (b^i)^{1/\beta}} \left[\sum \frac{b^k}{q^{1/\beta} - (b^k)^{1/\beta}} - \frac{\beta+1}{\beta} \frac{f'(B^0)}{f''(B^0)} \right]^{-1}.$$

The computational result is $f'''(B^0) = 0$. Since B^0 is a *momentum* density (see (56)), one can choose $f(B^0) = \gamma(B^0)^2$. Thus, the *linear* Hamiltonian density $\mathbf{H}_1 = B^1 + \gamma(B^0)^2$ determines the integrable hydrodynamic chain

$$B_{t^1}^k = B_x^{k+1} + \frac{2\beta\gamma}{\beta+1} B^0 B_x^k + 2\gamma \frac{k+2\beta}{1+\beta} B^k B_x^0, \quad k = 0, 1, \dots, \quad (62)$$

whose generating function of conservation laws is given by (see (61))

$$q_{t^1} = \beta \partial_x (q^{1+1/\beta} + 2\gamma B^0 q). \quad (63)$$

This is exactly the Kupershmidt hydrodynamic chain (see [21]) up to scaling $B^k \rightarrow \frac{\beta+1}{2\gamma} B^k$.

6.2 The Miura type and reciprocal transformations

Let us consider two hydrodynamic chains

$$C_t^k = \sum_{n=0}^{k+1} F_n^k(\mathbf{C}) C_x^n, \quad B_t^k = \sum_{n=0}^{k+1} G_n^k(\mathbf{B}) B_x^n, \quad k = 0, 1, 2, \dots,$$

related by the infinitely many transformations

$$B^k = B^k(C^0, C^1, \dots, C^{k+1}), \quad k = 0, 1, 2, \dots \quad (64)$$

Definition 3: The transformations (64) connecting *B-hydrodynamic chain* (right) and **modified C-hydrodynamic chain** (left) are said to be the **Miura type transformations**.

Theorem 5: The hydrodynamic chain (42) is connected with the Kupershmidt hydrodynamic chain

$$B_y^k = B_z^{k+1} + (\gamma - 1) B^0 B_z^k + (k + 2\gamma) B^k B_z^0, \quad k = 0, 1, 2, \dots \quad (65)$$

by the reciprocal transformation

$$dz = \mathbf{H}_0(C^0) dx + \gamma \mathbf{H}_0'^2(C^0) C^1 dt^1, \quad dy^1 = \frac{\gamma(1-\gamma)}{1-2\gamma} dt^1,$$

where $(1-2\gamma)B^k = C^{k+1}(C^0)^{-1+(k+1)/(1-2\gamma)}$ (see (64)).

Proof: Can be obtained by a straightforward calculation.

Remark: The generating function of conservation laws (50) transforms in the generating function of conservation laws for the Kupershmidt hydrodynamic chains (65)

$$q_y = \frac{\gamma-1}{\gamma} \partial_z [(1-2\gamma) q^{\frac{\gamma}{\gamma-1}} + \gamma B^0 q],$$

where the *generating function of the Miura type transformations* is $s = \mathbf{H}_0(C^0)q$.

Thus, if hydrodynamic reductions of the Kupershmidt hydrodynamic chains (M -dMKP hierarchy, see [2] and [26]) are known, then they can be recalculated to hydrodynamic reductions of M -dDym hierarchy (see also details in [34]) by the above reciprocal transformation.

The hydrodynamic chains (62), (65) are particular cases of the Kupershmidt hydrodynamic chain (see [21])

$$B_t^k = B_x^{k+1} + \frac{1}{\varepsilon} B^0 B_x^k + (k + \delta) B^k B_x^0, \quad k = 0, 1, 2, \dots, \quad (66)$$

which are equivalent to each other under an invertible transformation for any fixed value of the parameter ε (see the theorem below).

Theorem 6: *The Gibbons equation*

$$\lambda_t - (p^\varepsilon + \frac{B^0}{\varepsilon})\lambda_x = \frac{\partial \lambda}{\partial p} \left[p_t - \partial_x \left(\frac{p^{\varepsilon+1}}{\varepsilon + 1} + \frac{B^0}{\varepsilon} p \right) \right] \quad (67)$$

describes a deformation of the Riemann mapping

$$\lambda = p^{\varepsilon(1-\delta)} + (1 - \delta) \sum_{k=0}^{\infty} B^k p^{-\varepsilon(k+\delta)},$$

where the coefficients B^k satisfy the Kupershmidt hydrodynamic chain (66).

6.3 Reciprocal transformations and generating functions of conservation laws

The classification of integrable hydrodynamic chains associated with the Kupershmidt Poisson brackets and determined by the *quasilinear* and *fully nonlinear* Hamiltonian densities will be given in separate publication. In this sub-section we restrict our consideration on corresponding generating functions of conservation laws.

Theorem 7: *In the **quasilinear** case the generating function of conservation laws (see (60))*

$$q_{t^1} = \beta \partial_x (g(B^0) q^{1+1/\beta} + [g'(B^0) B^1 + f'(B^0)] q) \quad (68)$$

*can be reduced to the **linear** case iff*

$$g(B^0) = (B^0 + \delta)^{-(1+1/\beta)}.$$

Proof: An arbitrary reciprocal transformation

$$dz = F dx + G dt, \quad dy = dt \quad (69)$$

reduces (68) to

$$p_{y^1} = \beta \partial_z (g(B^0) F^{1+1/\beta} p^{1+1/\beta} + [(g'(B^0) B^1 + f'(B^0)) F - G/\beta] p), \quad (70)$$

where the generating function of the Miura type transformations is $q = Fp$. Since in the *linear* case (see (63))

$$p_{y^1} = \beta \partial_z \left(q^{1+1/\beta} + \tilde{B}^0 p \right), \quad (71)$$

then $g(B^0)F^{1+1/\beta} = 1$, i.e. the function F must be a some function of B^0 . However, F is a conservation law density, B^0 is a conservation law density (see (56)), then F is a *linear* function with respect to B^0 . Thus, indeed, $g(B^0) = (B^0 + \delta)^{-(1+1/\beta)}$, where δ is an arbitrary constant.

The first Miura type transformation is (cf. (70) and (71))

$$\tilde{B}^0 = -2\delta \frac{\beta+1}{\beta} (B^0 + \delta)^{-(2+1/\beta)} B^1 + (\delta - B^0) f'(B^0) + f(B^0).$$

Since \tilde{B}^0 is a conservation law density (see (63)) in the independent variables z, y , the r.h.s. of the expression

$$\tilde{B}^0(B^0 + \delta) = -2\delta \frac{\beta+1}{\beta} (B^0 + \delta)^{-(1+1/\beta)} B^1 + [(\delta - B^0) f'(B^0) + f(B^0)] (B^0 + \delta)$$

is a conservation law density in the independent variables x, t . However, this r.h.s. must coincide with the *quasilinear* Hamiltonian density $\mathbf{H}_1 = g(B^0)B^1 + f(B^0)$ up to the momentum density B^0 and an arbitrary constant ε . Thus, we have the ODE of the first order

$$-2\delta \frac{\beta+1}{\beta} f(B^0) = [(\delta - B^0) f'(B^0) + f(B^0)] (B^0 + \delta),$$

whose solution is

$$f(B^0) = (B^0 - \delta)^{2+1/\beta} (B^0 + \delta)^{-(1+1/\beta)}.$$

Then the *quasilinear* Hamiltonian density (ξ is an arbitrary constant)

$$\mathbf{H}_1 = (B^0 + \delta)^{-(1+1/\beta)} [B^1 + \xi (B^0 - \delta)^{2+1/\beta}]$$

creates an integrable hydrodynamic chain reducible to the Kupershmidt hydrodynamic chain.

Theorem 8: *In the **fully nonlinear** case the generating function of conservation laws*

$$q_{t^1} = \beta \partial_x \left(q^{1+1/\beta} \frac{\partial \mathbf{H}_1}{\partial B^1} + q \frac{\partial \mathbf{H}_1}{\partial B^0} \right) \quad (72)$$

*can be reduced to the **linear** case iff*

$$\mathbf{H}_1 = [B^1 + \varphi(B^0)]^{\frac{\beta}{2\beta+1}}, \quad (73)$$

where $\varphi(B^0)$ is a some function.

Proof: An arbitrary reciprocal transformation (69) reduces (72) to

$$p_{y^1} = \beta \partial_z \left[F^{1+1/\beta} \frac{\partial \mathbf{H}_1}{\partial B^1} p^{1+1/\beta} + \left(F \frac{\partial \mathbf{H}_1}{\partial B^0} - \frac{G}{\beta} \right) p \right].$$

This generating function coincides with the *linear* case if $F^{1+1/\beta} \partial \mathbf{H}_1 / \partial B^1 = 1$. Since \mathbf{H}_1 depends on B^0 and B^1 , then the conservation law density $F(B^0, B^1)$ must be a *linear* function with respect to the Hamiltonian density \mathbf{H}_1 up to the momentum density B^0 and an arbitrary constant δ , which can be removed by shift $z \rightarrow z + \varepsilon y$, where ε is an appropriate constant. Thus, we have the ODE

$$(\mathbf{H}_1)^{1+1/\beta} \frac{\partial \mathbf{H}_1}{\partial B^1} = 1,$$

whose solution is (73) up to an insufficient factor. The function $\varphi(B^0)$ can be determined in the same way as in the *quasilinear* case, but we avoid this complicated computation in this paper. Further details can be found in [11]. Nevertheless, we would like to emphasize that an application of reciprocal transformations to the quasilinear case and moreover to fully nonlinear case significantly simplifies computations made by the Hamiltonian hydrodynamic reductions method (see the beginning of this section).

7 Hydrodynamic chains determined by the Kupershmidt–Manin bracket

In this paper we restrict our consideration on the *linear* case $\mathbf{H}_2 = A^2/2 + f(A^0, A^1)$ in general. The hydrodynamic chain (34)

$$A_t^k = A_x^{k+1} + \frac{\partial f}{\partial A^1} A_x^k + (k+1) A^k \left(\frac{\partial f}{\partial A^1} \right)_x + k A^{k-1} \left(\frac{\partial f}{\partial A^0} \right)_x. \quad (74)$$

creates the generating function of conservation laws (36)

$$p_t = \partial_x \left(\frac{p^2}{2} + \frac{\partial f}{\partial A^1} p + \frac{\partial f}{\partial A^0} \right), \quad (75)$$

where p is a some function of all moments A^0, A^1, A^2, \dots and “spectral parameter” λ , which is not introduced yet. If $p \rightarrow \infty$, the corresponding flux of the generating function of conservation laws (75) also tends to infinity. One can seek p as the Laurent series in the form (38)

$$p = \lambda - \tilde{\mathbf{H}}_{-1} - \frac{\tilde{\mathbf{H}}_0}{\lambda} - \frac{\tilde{\mathbf{H}}_1}{\lambda^2} - \frac{\tilde{\mathbf{H}}_2}{\lambda^3} - \dots \quad (76)$$

If hydrodynamic chain (74) is integrable, then the *infinite* series of conservation laws

$$\begin{aligned} \partial_t \tilde{\mathbf{H}}_{-1} &= \partial_x \left(\tilde{\mathbf{H}}_0 + \frac{1}{2} \tilde{\mathbf{H}}_{-1}^2 - \frac{\partial f}{\partial A^0} \right), & \partial_t \tilde{\mathbf{H}}_0 &= \partial_x \tilde{\mathbf{H}}_1, \\ \partial_t \tilde{\mathbf{H}}_k &= \partial_x \left(\tilde{\mathbf{H}}_{k+1} - \frac{1}{2} \sum_{m=0}^{k-1} \tilde{\mathbf{H}}_m \tilde{\mathbf{H}}_{k-1-m} \right), \end{aligned} \quad (77)$$

where

$$\tilde{\mathbf{H}}_{-1} = \frac{\partial f}{\partial A^1},$$

can be derived by the substitution this series (76) in (75).

Since $\partial f/\partial A^1$ is a conservation law density, then $\partial f/\partial A^1$ must be a *linear* function of conservation law densities A^0 and A^1 (the Casimir density and the momentum density of the Kupershmidt–Manin bracket) only. Thus,

$$f = \alpha A^0 A^1 + \beta (A^1)^2 + \varphi(A^0),$$

where α and β are arbitrary constants (a linear term proportional A^1 is removed by a linear change of the independent variable $x \rightarrow x + t \text{ const}$), $\varphi(A^0)$ is not determined function yet (if $\beta \neq 0$, the constant α is unessential and can be removed by the shift $\alpha A^0 + 2\beta A^1 \rightarrow 2\beta A^1$, see [10] and [27]).

Let us consider sub-cases depending on a different number of such constants.

7.1 The Benney hydrodynamic chain

In the simplest case $\tilde{\mathbf{H}}_{-1} = \text{const}$, then the generating function of conservation laws (75) is (the central term $\mu \text{ const}$ can be removed by a linear change of the independent variable $x \rightarrow x + t \text{ const}$)

$$\mu_t = \partial_x \left(\frac{\mu^2}{2} + f'(A^0) \right).$$

If the hydrodynamic chain (74)

$$A_t^k = A_x^{k+1} + k A^{k-1} f''(A^0) A_x^0$$

is integrable, then a substitution of the Laurent series (76)

$$\mu = \lambda - \frac{\mathbf{H}_0}{\lambda} - \frac{\mathbf{H}_1}{\lambda^2} - \frac{\mathbf{H}_2}{\lambda^3} - \dots$$

yields the infinite series of conservation laws (77)

$$\partial_t \mathbf{H}_0 = \partial_x \mathbf{H}_1, \quad \partial_t \mathbf{H}_k = \partial_x \left(\mathbf{H}_{k+1} - \frac{1}{2} \sum_{m=0}^{k-1} \mathbf{H}_m \mathbf{H}_{k-1-m} \right),$$

where

$$\mathbf{H}_0 = f'(A^0).$$

Since $f'(A^0)$ is a conservation law density, then $f'(A^0)$ must be a *linear* function of the conservation law density A^0 . Thus, one can choose $f(A^0) = (A^0)^2/2$ up to unessential constants, which can be removed by the Galilean transformation, a shift and a scaling. Thus, the first classification result is precisely the Benney hydrodynamic chain (2).

7.2 The *mixed* modified Benney hydrodynamic chain

Let us consider the most general case when all constants are not vanished. In this case we replace the notation $A^k \rightarrow E^k$. Taking into account that

$$E_t^0 = \partial_x \left(E^1 + E^0 \frac{\partial f}{\partial E^1} \right), \quad E_t^1 = \partial_x \left[E^2 + 2E^1 \frac{\partial f}{\partial E^1} + E^0 \frac{\partial f}{\partial E^0} - f \right],$$

one can obtain

$$\tilde{\mathbf{H}}_0 = 2\beta E^2 + 4\beta^2(E^1)^2 + 4\alpha\beta E^0 E^1 + \frac{\alpha^2}{2}(E^0)^2 + 2\alpha E^1 + (1 + 2\beta E^0)\varphi' - 2\beta\varphi$$

from the first conservation law (77). Since the Hamiltonian density is $\mathbf{H}_2 = E^2/2 + \alpha E^0 E^1 + \beta(E^1)^2 + \varphi(E^0)$, then the conservation law density $\tilde{\mathbf{H}}_0 - 4\beta\mathbf{H}_2$ must be a *linear* function of E^0 and E^1 only. It means, that $\tilde{\mathbf{H}}_0$ exists iff the function φ satisfies the ODE of the first order

$$(1 + 2\beta E^0)\varphi' - 6\beta\varphi + \frac{\alpha^2}{2}(E^0)^2 + \gamma E^0 + \delta = 0, \quad (78)$$

where γ and δ are arbitrary constants. Integrating this equation, one can obtain the Hamiltonian

$$\bar{\mathbf{H}}_2 = \int \left[\frac{E^2}{2} + \alpha E^0 E^1 + \beta(E^1)^2 - 2\frac{\beta^2 \varepsilon^2}{3}(E^0)^3 + \left(\frac{\alpha^2}{4\beta} - \beta\varepsilon^2\right)(E^0)^2 \right] dx,$$

where ε is an arbitrary constant.

Thus, the most general (*mixed modified* Benney) hydrodynamic chain

$$E_t^k = E_x^{k+1} + (\alpha E^0 + 2\beta E^1)E_x^k + (k+1)E^k(\alpha E^0 + 2\beta E^1)_x + kE^{k-1}(\alpha E^1 - 2\beta^2 \varepsilon^2 (E^0)^2 + \left(\frac{\alpha^2}{2\beta} - 2\beta\varepsilon^2\right)E^0)_x.$$

determined by the Kupershmidt–Manin bracket and the *linear* Hamiltonian has the generating function of conservation laws (instead of p we use s for this case)

$$s_t = \partial_x \left(\frac{s^2}{2} + (\alpha E^0 + 2\beta E^1)s + \alpha E^1 - 2\beta^2 \varepsilon^2 (E^0)^2 + \left(\frac{\alpha^2}{2\beta} - 2\beta\varepsilon^2\right)E^0 \right).$$

Remark: *The substitution*

$$\mu = s + \alpha E^0 + 2\beta E^1$$

connects the above generating function of conservation laws with the generating function of conservation laws

$$\mu_t = \partial_x \left(\frac{\mu^2}{2} + A^0 \right) \quad (79)$$

for the Benney hydrodynamic chain (2).

If $\varepsilon = 0$, then the above hydrodynamic chain reduces to the *twice modified* Benney hydrodynamic chain

$$C_t^k = C_x^{k+1} + (\alpha C^0 + 2\beta C^1)C_x^k + (k+1)C^k(\alpha C^0 + 2\beta C^1)_x + kC^{k-1}(\alpha C^1 + \frac{\alpha^2}{2\beta}C^0)_x$$

connected with the generating function of conservation laws (instead of p we use q for this case)

$$q_t = \partial_x \left(\frac{q^2}{2} + (\alpha C^0 + 2\beta C^1)q + \alpha C^1 + \frac{\alpha^2}{2\beta}C^0 \right).$$

Remark: *The substitution*

$$\mu = q + \alpha C^0 + 2\beta C^1$$

connects the above generating function of conservation laws with the generating function of conservation laws (79) for the Benney hydrodynamic chain (2), where

$$\tilde{\mathbf{H}}_0 = 2\beta C^2 + 2\alpha C^1 + \frac{\alpha^2}{2\beta} C^0 + \alpha^2 (C^0)^2 + 4\alpha\beta C^0 C^1 + 4\beta^2 (C^1)^2.$$

If $\beta = 0$, then the Hamiltonian (see (78))

$$\bar{\mathbf{H}}_2 = \int \left(\frac{B^2}{2} + \alpha B^0 B^1 - \frac{\alpha^2}{6} (B^0)^3 - \frac{\gamma}{2} (B^0)^2 \right) dx$$

determines the *modified* Benney hydrodynamic chain (the constant α can be removed by scaling $\alpha B^k \rightarrow B^k$, $\alpha \neq 0$)

$$B_t^k = B_x^{k+1} + \alpha B^0 B_x^k + \alpha(k+1) B^k B_x^0 + k B^{k-1} \left(\alpha B^1 - \frac{\alpha^2}{2} (B^0)^2 - \gamma B^0 \right)_x$$

connected with the generating function of conservation laws

$$p_t = \partial_x \left(\frac{p^2}{2} + \alpha B^0 p + \alpha B^1 - \frac{\alpha^2}{2} (B^0)^2 - \gamma B^0 \right).$$

Remark: *The substitution*

$$p = \mu - \alpha B^0$$

connects the above generating function of conservation laws with the generating function of conservation laws

$$\mu_t = \partial_x \left(\frac{\mu^2}{2} + A^0 \right)$$

for the Benney hydrodynamic chain (2), where $A^0 = 2\alpha B^1 - \gamma B^0$.

7.3 Reciprocal transformations and non-linear cases

In comparison with the previous section the *quasilinear* case $\mathbf{H}_2 = g(A^0, A^1)A^2 + f(A^0, A^1)$ has *two* sub-cases. It is easy to see by an application of the reciprocal transformation (69) to the generating function of conservation laws (36)

$$p_t = \partial_x (g(A^0, A^1) p^2 + \text{lower order terms}).$$

The modified generating function of conservation laws is

$$\tilde{p}_y = \partial_z (\tilde{p}^2 + \text{lower order terms}),$$

where

$$g(A^0, A^1) F^2 = 1.$$

If g depends on A^0 only, then the conservation law density F is a *linear* function of the Casimir density A^0 . Thus, $g = (A^0 + \delta)^{-2}$. If g depends on the both moments A^0 and A^1 , then the conservation law density F is a *linear* function of the Casimir and the momentum densities A^0 and A^1 . Thus, $g = (A^1 + \delta)^{-2}$ (a linear term proportional A^0 is removed by a linear change of the independent variable $x \rightarrow x + t \text{ const}$; see [10]).

The *fully nonlinear* case is reducible to the *linear* case under the reciprocal transformation (69) if

$$\frac{\partial \mathbf{H}_2}{\partial A^2} F^2 = 1.$$

Since F is a conservation law density depending on all three moments A^0, A^1, A^2 , then F must be a *linear* function of the Hamiltonian density \mathbf{H}_2 up to the momentum and the Casimir densities, which can be removed by a shift $x \rightarrow x + \varepsilon t$, where ε is an appropriate constant. Thus, the Hamiltonian density

$$\mathbf{H}_2 = [A^2 + f(A^0, A^1)]^{1/3}$$

determines an integrable hydrodynamic chain. The function $f(A^0, A^1)$ can be found by the method of the Hamiltonian hydrodynamic reductions. Further details can be found in [12]. Nevertheless, we would like to emphasize that the “asymptotic analysis” (see the beginning of this section) significantly simplifies computations made by the Hamiltonian hydrodynamic reductions method.

8 Non-Hamiltonian hydrodynamic chains

The hydrodynamic chains associated with the Kupershmidt–Manin Poisson brackets (see (20) and (21))

$$A_{t^m}^k = [kA^{k+n-1}\partial_x + n\partial_x A^{k+n-1}] \frac{\delta \bar{\mathbf{H}}_{m+1}}{\delta A^n}$$

possess the “waterbag moment decomposition” (15), and the corresponding hydrodynamic reductions are

$$a_t^i = \partial_x \left[\sum_{n=0}^{m+1} (a^i)^n \frac{\delta \bar{\mathbf{H}}_m}{\delta A^n} \right],$$

where the Hamiltonian $\bar{\mathbf{H}}_{m+1} = \int \mathbf{H}_{m+1}(A^0, A^1, \dots, A^m, A^{m+1}) dx$. The hydrodynamic chains associated with the Kupershmidt Poisson brackets (58) also possess the waterbag moment decomposition (57), and the corresponding hydrodynamic reductions are given by (59).

In the both above cases corresponding hydrodynamic chains *linearly* depend on the discrete variable k .

The Hamiltonian approach presented in this paper is useful for non-Hamiltonian hydrodynamic chains too. For instance, the *deformed* Benney hydrodynamic chain (see [10], cf. (2))

$$A_t^k = A_x^{k+1} + \varepsilon A^0 A_x^k + k A^{k-1} A_x^0, \quad k = 0, 1, \dots \quad (80)$$

is not Hamiltonian, but still is integrable (at least the Hamiltonian formalism for the above hydrodynamic chain is unknown). Nevertheless, this hydrodynamic chain has the same moment decomposition (15)

$$A^n = \frac{1}{n+1} \sum \varepsilon_k (u^k)^{n+1}, \quad (81)$$

(where $\sum \varepsilon_m = 0$, cf. (48)) as the Benney hydrodynamic chain (2), but a corresponding generating function of conservation laws is little bit more complicated. The corresponding hydrodynamic type system

$$u_t^i = u^i u_x^i + \varepsilon A^0 u_x^i + A_x^0 \quad (82)$$

can be written in the conservative form

$$a_t^i = \partial_x \left[\left(\frac{1}{\varepsilon} (\ln a^i - 1) + \varepsilon A^0 \right) a^i \right], \quad (83)$$

where

$$a^i = \exp(\varepsilon u^i). \quad (84)$$

Following the Hamiltonian approach, the deformed Benney hydrodynamic chain has the generating function of conservation laws

$$p_t = \partial_x \left[\left(\frac{1}{\varepsilon} (\ln p - 1) + \varepsilon A^0 \right) p \right], \quad (85)$$

which reduces to the first equation in (4) by $\varepsilon \rightarrow 0$ under the substitution

$$p = \exp(\varepsilon \mu).$$

Theorem 9: *The deformed Benney hydrodynamic chain (80) satisfies the Gibbons equation (cf. (8))*

$$\lambda_t - \left(\frac{1}{\varepsilon} \ln p + \varepsilon A^0 \right) \lambda_x = \frac{\partial \lambda}{\partial p} \left(p_t - \partial_x \left[\left(\frac{1}{\varepsilon} (\ln p - 1) + \varepsilon A^0 \right) p \right] \right),$$

where the equation of the Riemann mapping is (7).

Proof: can be obtained by a straightforward substitution (7) and (80) into the above Gibbons equation.

Corollary: The substitution (81) in the equation of the Riemann mapping (7) yields again the equation of the Riemann surface (14)

$$\lambda = \mu - \sum \varepsilon_k \ln(\mu - u^k).$$

Remark: Rational (Zakharov) reductions (of the Benney hydrodynamic chain, see [37]) can be obtained by a limit from the waterbag reductions (see, for instance, [3]): one can substitute the series $\tilde{u}^{(k)} = u^k + v^k/\varepsilon^k + w^k/(\varepsilon^k)^2 + \dots$, $\varepsilon^k \rightarrow \infty$ to the particular case of the waterbag reduction

$$\lambda = \mu - \sum_{k=1}^N \varepsilon_k \ln \frac{\mu - \tilde{u}^k}{\mu - u^k}.$$

The same result can be obtained by a straightforward substitution the corresponding moment decomposition (17)

$$A^k = \sum_{n=1}^N (u^n)^k v^n \quad (86)$$

to the deformed Benney hydrodynamic chain. The corresponding hydrodynamic type system (cf. the second equation below with (82))

$$v_t^k = \partial_x(u^k v^k) + \varepsilon A^0 v_x^k, \quad u_t^k = \partial_x \left(\frac{(u^k)^2}{2} + A^0 \right) + \varepsilon A^0 u_x^k$$

can be written in the conservative form (can be obtained directly from (85) by substitution series $p^{(k)} \rightarrow a^k + b^k \lambda + \dots$; cf. (83) and the first equation from the system below)

$$a_t^i = \partial_x \left[\left(\frac{1}{\varepsilon} (\ln a^i - 1) + \varepsilon A^0 \right) a^i \right], \quad b_t^i = \partial_x \left[\left(\frac{1}{\varepsilon} \ln a^i + \varepsilon A^0 \right) b^i \right],$$

where $a^i = \exp(\varepsilon u^i)$ (cf. (84)) and $b^i = \varepsilon v^i \exp(\varepsilon u^i)$.

9 The Zakharov hydrodynamic reductions

Let us consider the integrable hydrodynamic chain (see [21])

$$C_t^k = C_x^{k+1} + C^0 C_x^k + (\alpha k + \beta) C^k C_x^0 + \gamma k C^{k-1} [C^1 + \frac{\beta - \alpha - \gamma}{2} (C^0)^2]_x, \quad k = 0, 1, 2, \dots$$

This chain does not possess waterbag hydrodynamic reductions given by the moment decomposition (81). However, the Zakharov hydrodynamic reductions (86) exist if a hydrodynamic chain has a *linear* dependence with respect to the discrete variable k . Thus, the corresponding hydrodynamic reduction is

$$u_t^i = u^i u_x^i + C^0 u_x^i + \alpha u^i C_x^0 + \gamma [C^1 + \frac{\beta - \alpha - \gamma}{2} (C^0)^2]_x,$$

$$v_t^i = u^i v_x^i + v^i u_x^i + C^0 v_x^i + \beta v^i C_x^0,$$

where $C^k = \Sigma (u^i)^k v^i$. Introducing the new field variables

$$a^i = (u^i - \gamma C^0)^{\frac{\gamma+1}{\gamma+\alpha}},$$

the *first* N equations can be written in the conservative form

$$a_t^i = \partial_x \left(\frac{\gamma+1}{2\gamma+\alpha+1} (a^i)^{\frac{2\gamma+\alpha+1}{\gamma+1}} + (\gamma+1) C^0 a^i \right).$$

Thus, the generating function of conservation laws

$$p_t = \partial_x \left(\frac{\gamma+1}{2\gamma+\alpha+1} p^{\frac{2\gamma+\alpha+1}{\gamma+1}} + (\gamma+1) C^0 p \right) \quad (87)$$

is obtained by the replacement $a^i \rightarrow p$. Substituting the Taylor expansion $p^{(k)} \rightarrow a^k + \lambda b^k + \dots$, where λ is a local parameter, one obtains the *second* N conservation laws

$$b_t^i = \partial_x \left[\left((a^i)^{\frac{\gamma+\alpha}{\gamma+1}} + (\gamma+1)C^0 \right) b^i \right],$$

where $b^i = (a^i)^{\frac{1-\beta}{\gamma+1}} v^i$.

Remark: The Kupershmidt hydrodynamic chain (66) (see [21]) has the generating function of conservation laws (see (67))

$$p_t = \partial_x \left(\frac{p^{\varepsilon+1}}{\varepsilon+1} + \frac{B^0}{\varepsilon} p \right). \quad (88)$$

Thus, (comparing (87) and (88)) we conclude that both above hydrodynamic chains are equivalent under invertible transformations $B^k = B^k(C^0, C^1, \dots, C^k)$, $k = 0, 1, 2, \dots$, where

$$\varepsilon = \frac{\gamma + \alpha}{\gamma + 1}, \quad \delta = \frac{\beta + \gamma}{\gamma + \alpha}.$$

Explicit expressions can be found by the comparison two series

$$\sum_{k=0}^{\infty} \frac{B^k}{p^{k+1}} = (\gamma + \alpha) \sum_{k=0}^{\infty} \frac{C^k}{(p + \gamma C^0)^{k+1}},$$

where the equation of the Riemann surface is given by

$$\lambda = p^{\frac{\alpha-\beta}{\gamma+\alpha}} \left[1 + (\alpha - \beta) \sum_{k=0}^{\infty} \frac{C^k}{(p + \gamma C^0)^{k+1}} \right].$$

Then the moments B^k can be written explicitly via the field variables a^i and b^i in the form (cf. (17))

$$B^k = (\gamma + \alpha) \sum_{i=1}^N (a^i)^{\frac{(\gamma+\alpha)k+\beta-1}{\gamma+1}} b^i.$$

10 Conclusion

At this moment several very powerful approaches are appropriate in the theory of classification of integrable hydrodynamic chains. These are the symmetry approach (see [27]), the method of hydrodynamic reductions (see [7], [16]), the method of pseudo-potentials (see [7], [36]), the tensor approach ([10]).

However, *this* Hamiltonian approach (based on the concept of the Hamiltonian hydrodynamic reductions) in most cases allows to check integrability and classify integrable hydrodynamic chains avoiding symbolic software. Moreover, this method simultaneously allows to construct infinitely many particular solutions parameterized by arbitrary constants in the explicit form.

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References

- [1] *D.J. Benney*, Some properties of long non-linear waves, *Stud. Appl. Math.*, **52** (1973) 45-50.
- [2] *M. Blaszk, B.M. Szablikowski*, Classical R -matrix theory of dispersionless systems: I. (1+1)-dimension theory, *J. Phys. A: Math. Gen.* **35** (2002) 10325-10344. *M. Blaszk, B.M. Szablikowski*, Classical R -matrix theory of dispersionless systems: II. (2+1)-dimension theory, *J. Phys. A: Math. Gen.* **35** (2002) 10345-10364.
- [3] *L.V. Bogdanov, B.G. Konopelchenko*, Symmetry constraints for dispersionless integrable equations and systems of hydrodynamic type, *Phys. Lett. A*, **330** (2004) 448-459.
- [4] *I.Ya. Dorfman*, Dirac structures and integrability of nonlinear evolution equations; Nonlinear Science: Theory and Applications, John Wiley & Sons, New York (1993) 176 pp.
- [5] *B.A. Dubrovin, S.P. Novikov*, Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov-Whitham averaging method, *Soviet Math. Dokl.*, **27** (1983) 665-669. *B.A. Dubrovin, S.P. Novikov*, Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, *Russian Math. Surveys*, **44** No. 6 (1989) 35-124.
- [6] *E.V. Ferapontov*, Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, *Amer. Math. Soc. Transl. (2)*, **170** (1995) 33-58.
- [7] *E.V. Ferapontov, K.R. Khusnutdinova*, On integrability of (2+1)-dimensional quasilinear systems, *Comm. Math. Phys.* **248** (2004) 187-206, *E.V. Ferapontov, K.R. Khusnutdinova*, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, *J. Phys. A: Math. Gen.* **37** No. 8 (2004) 2949 - 2963.
- [8] *E.V. Ferapontov, K.R. Khusnutdinova, M.V. Pavlov*, Classification of integrable (2+1) dimensional quasilinear hierarchies, *Theor. Math. Phys.* **144** No. 1 (2005) 35-43.
- [9] *E.V. Ferapontov, K.R. Khusnutdinova, S.P. Tsarev*, On a class of three dimensional integrable Lagrangians, *Comm. Math. Phys.*

- [10] *E.V. Ferapontov, D.G. Marshall*, Differential-geometric approach to the integrability of hydrodynamic chains: the Haantjes tensor, *arXiv:nlin.SI/0505013*.
- [11] *E.V. Ferapontov, D.G. Marshall, M.V. Pavlov*, The Kupershmidt Poisson brackets and classification of associated integrable hydrodynamic chains.
- [12] *E.V. Ferapontov, D.G. Marshall, M.V. Pavlov*, The Kupershmidt–Manin Poisson bracket and classification of associated integrable hydrodynamic chains.
- [13] *E.V. Ferapontov, O.I. Mokhov*, Nonlocal Hamiltonian operators of hydrodynamic type that are connected with metrics of constant curvature, *Russian Math. Surveys*, **45** No. 3 (1990) 218–219.
- [14] *E.V. Ferapontov, M.V. Pavlov*, Hydrodynamic reductions of the heavenly equation, *Class. Quantum Grav.* **20** (2003) 2429–2441.
- [15] *J. Gibbons*, Collisionless Boltzmann equations and integrable moment equations, *Physica D*, **3** (1981) 503–511.
- [16] *J. Gibbons, S.P. Tsarev*, Reductions of Benney’s equations, *Phys. Lett. A*, **211**, (1996) 19–24. *J. Gibbons, S.P. Tsarev*, Conformal maps and reductions of the Benney equations, *Phys. Lett. A*, **258** (1999) 263–270.
- [17] *J. Gibbons, L.A. Yu*, The initial value problem for reductions of the Benney equations, *Inverse Problems* **16** No. 3 (2000) 605–618, *L.A. Yu*, Waterbag reductions of the dispersionless discrete KP hierarchy, *J. Phys. A: Math. Gen.*, **33** (2000) 8127–8138.
- [18] *Yu. Kodama*, Solution of the dispersionless Toda equation, *Phys. Lett. A*, **147** (1990) 477–480.
- [19] *Yu. Kodama*, A method for solving the dispersionless KP equation and its exact solutions. *Phys. Lett. A*, **129** No. 4 (1988) 223–226.
- [20] *I.M. Krichever*, The dispersionless equations and topological minimal models, *Comm. Math. Phys.*, **143** No. 2 (1992) 415–429. *I.M. Krichever*, The τ -function of the universal Whitham hierarchy, matrix models and topological field theories, *Comm. Pure Appl. Math.* **47** (1994) 437–475.
- [21] *B.A. Kupershmidt*, Deformations of integrable systems, *Proc. Roy. Irish Acad. Sect. A* **83** No. 1 (1983) 45–74. *B.A. Kupershmidt*, Normal and universal forms in integrable hydrodynamical systems, *Proceedings of the Berkeley-Ames conference on nonlinear problems in control and fluid dynamics* (Berkeley, Calif., 1983), in *Lie Groups: Hist., Frontiers and Appl. Ser. B: Systems Inform. Control, II*, Math Sci Press, Brookline, MA, (1984) 357–378.
- [22] *B.A. Kupershmidt, Yu.I. Manin*, Long wave equations with a free surface. II. The Hamiltonian structure and the higher equations, *Func. Anal. Appl.* **12** No. 1 (1978) 25–37. *D.R. Lebedev, Yu.I. Manin*, Conservation laws and representation of Benney’s long wave equations, *Phys. Lett. A*, **74** No. 3,4 (1979) 154–156.

- [23] *B.A. Kupershmidt*, Hydrodynamic chains of Pavlov’s class, accepted in Phys. Lett. A.
- [24] *A.Ya. Maltsev, S.P. Novikov*, On the local systems Hamiltonian in the weakly non-local Poisson brackets, Physica D **156** (2001) 53-80.
- [25] *M. Manas, L.M. Alonso, E. Medina*, Reductions of the dispersionless KP hierarchy, Theor. Math. Phys. **133** No. 3 (2002) 1712-1721, *B. Konopelchenko, L.M. Alonso, E. Medina*, Quasiconformal mappings and solutions of the dispersionless KP hierarchy, Theor. Math. Phys. **133** No. 2 (2002) 1529-1538.
- [26] *M. Manas*, S -functions, reductions and hodograph solutions of the r th dispersionless modified KP and Dym hierarchies, J. Phys. A: Math. Gen., **37** (2004) 11191–11221.
- [27] *M.V. Pavlov*, Classification of the Egorov hydrodynamic chains, Theor. Math. Phys. **138** No. 1 (2004) 55-71.
- [28] *M.V. Pavlov*, Integrable hydrodynamic chains, J. Math. Phys. **44** No. 9 (2003) 4134-4156.
- [29] *M.V. Pavlov*, The Kupershmidt hydrodynamic chains.
- [30] *M.V. Pavlov*, Hydrodynamic chains and a classification of their Poisson brackets.
- [31] *M.V. Pavlov*, Algebro-geometric approach in the theory of integrable hydrodynamic type systems.
- [32] *M.V. Pavlov*, Classification of integrable hydrodynamic chains and generating functions of conservation laws.
- [33] *M.V. Pavlov*, Explicit solutions of the WDVV equation determined by “flat” hydrodynamic reductions of the dispersionless limit for KP, BKP–VN and Boyer–Finley equations.
- [34] *B.M. Szablikowski*, Gauge transformation and reciprocal link for (2+1) dimensional integrable field systems, J. Nonlin. Math. Phys., **13** (2006).
- [35] *S.P. Tsarev*, On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, Soviet Math. Dokl., **31** (1985) 488–491. *S.P. Tsarev*, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, Math. USSR Izvestiya **37** No. 2 (1991) 397–419.
- [36] *V.E. Zakharov*, Dispersionless limit of integrable systems in 2+1 dimensions, in Singular Limits of Dispersive Waves, Ed. N.M. Ercolani et al., Plenum Press, NY, (1994) 165-174.
- [37] *V.E. Zakharov*, Benney’s equations and quasi-classical approximation in the inverse problem method, Funct. Anal. Appl., **14** No. 2 (1980) 89-98. *V.E. Zakharov*, On the Benney’s Equations, Physica 3D (1981) 193-200.